Uniqueness results for quasilinear parabolic equations through viscosity solutions' methods

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Abstract

In this article, we are interested in uniqueness results for viscosity solutions of a general class of quasilinear, possibly degenerate, parabolic equations set in $I\!R^N$. Using classical viscosity solutions' methods, we obtain a general comparison result for solutions with polynomial growths but with a restriction on the growth of the initial data. The main application is the uniqueness of solutions for the mean curvature equation for graphs which was only known in the class of uniformly continuous functions. An application to the mean curvature flow is given.

Key-words: Quasilinear parabolic equations, solutions with polynomial growth, comparison results, mean curvature equation, viscosity solutions.
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1 Introduction

This article is a continuation of the program started in [3] (see [2] for an introductory paper) which aim is the study of the uniqueness properties for unbounded viscosity solutions of quasilinear, possibly degenerate, parabolic equations set in $I\!R^N$. This program was motivated by the following surprising result of Ecker and Huisken [12] : for any initial data $u_0 \in W^{1,\infty}_{\mathrm{loc}}(I\!R^N)$, there exists a smooth solution of the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \dfrac{\langle D^2 u Du, Du\rangle}{1+|Du|^2} = 0 \quad \text{in } I\!R^N \times (0,+\infty)\,, \\ u(x,0) = u_0(x) \quad \text{in } I\!R^N\,. \end{cases} \tag{1}$$

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Here and below the solution u is a real-valued functions, Du and D^2u denote respectively its gradient and Hessian matrix while $|\cdot|$ and $\langle \cdot, \cdot \rangle$ stand for the classical Euclidean norm and inner product in $I\!R^N$.

The very non-standard feature of this result is that no assumption on the behavior of the initial data at infinity is imposed and therefore the solutions may have also any possible behavior at infinity.

A natural question is then whether such a solution is unique or not. It is a very intriguing and challenging question since one has to take in account the lack of restriction on the behavior of the solutions at infinity, an unusual fact. As far as we know, this question in its full generality is still open in $I\!R^N$ for $N > 1$ while for $N = 1$, the result was proved independently and by different methods by Chou and Kwong [8] and Barles, Biton and Ley [4].

In a series of papers ([3], [4] and [5]), we address the more general question of the uniqueness of unbounded viscosity solutions, not only for (1) but also for more general quasilinear degenerate parabolic equations like

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{Tr}\left[b(x,t,Du)D^2u\right] + H(x,t,u,Du) = 0 \quad \text{in } I\!R^N \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } I\!R^N, \end{cases} \tag{2}$$

where $T > 0$ is any positive constant and b is a function taking values in the set of nonnegative symmetric matrix and $H \in C(I\!R \times [0,T] \times I\!R \times I\!R^N)$.

When one wants to prove such uniqueness results, the first key difficulty is the one we point out above i.e. the a priori unboundedness of solutions or more generally the fact that solutions may have any behavior at infinity. But the gradient dependence in the diffusion matrix b above is also a major difficulty: because of that, it does not seem possible to obtain uniqueness results through some kind of linearization procedure and even the uniqueness of smooth solutions is far from being obvious, except if one imposes restrictions on D^2u at infinity, a type of assumptions that we want to avoid. For results obtained by linearization procedure, we refer the reader for example to Crandall and Lions [10], Ishii [16] or Ley [18] where the uniqueness of solutions of first-order equations were obtained using "finite speed of propagation" type properties, to Barles [1] where optimal uniqueness results for solutions with exponential growth of (stationary) first-order equations were proved or to Barles, Buckdahn and Pardoux [6] for solutions with exponential growths of a system of Hamilton-Jacobi-Bellman type Equations.

To the best of our knowledge, the most general uniqueness results for quasilinear equations – i.e. for equations involving the above mentioned difficulty on the Du-dependence – concern only uniformly continuous viscosity solutions : we refer the reader the "Users'guide of viscosity solutions" of Crandall, Ishii and Lions [9] and to Giga, Goto, Ishii and Sato [15] for results in this direction.

The aim of this article is to push as far as possible the classical arguments used for proving comparison results for viscosity solutions in order to obtain such results for the largest possible class of quasilinear equations and initial data.

In the general comparison theorem we are able to prove by using this approach (see Theorem 2.1), the conditions we have to impose on the equation and the behavior of solution at infinity, namely to have a polynomial growth, seem rather reasonable. Surprisingly the main restriction concerns the initial data which has to satisfy the following, rather unnatural, condition: there exists a modulus of continuity $m^\dagger$ and $0 \leq \nu < (1+\sqrt{5})/2$ such that, for any $x, y \in I\!R^N$,

$$|u_0(x) - u_0(y)| \leq m\left((1+|x|+|y|)^\nu |x-y|\right) . \tag{3}$$

Unfortunately such type of restriction seems to be an unavoidable artefact of this method. We do not know how optimal is this result and in particular the limiting exponent $(1+\sqrt{5})/2$. In fact, since this result applies not only to the mean curvature equation (1) but also to a large class of degenerate and nondegenerate equations, we do not think that this exponent could have a geometrical interpretation. On the other hand, we believe that the result is true for any ν but we were unable to prove it.

Crandall and Lions [11] obtained related results under similar assumptions but for fully nonlinear pdes whose Hamiltonians depend only on the second derivatives of the solution u. Their techniques do not apply to our case. Comparing to the existence result for the mean curvature mentioned above, condition (3) may appear restrictive but we point out that our proof works under a general structure assumption on b and not only for the mean curvature equation. We refer to Section 2 for a precise statement of the assumptions on b and H and to Section 3 for a detailed treatment of the mean curvature equation for graphs.

Unfortunately the proof of this comparison result is very technical (see Section 4) and relies on the use of a tricky test-function that, as we already mentioned it above, we tried to build in an optimal way.

This result yields a comparison principle between semicontinuous sub- and supersolutions which is a key tool for obtaining existence results through the Perron's method (see [17], [9], etc.). It therefore allows us to show such an existence result of solutions with a suitable growth for (2).

The paper is divided as follows: in the next section, we start by setting the problem and the assumptions we will use. Then we state a comparison principle (Theorem 2.1) which is the main result of the paper. As an immediate consequence, we obtain the existence and uniqueness of a continuous viscosity solution to our problem (Corollary 2.1). We end this section with some examples of equations which are included in our study. Section 3 deals with the fundamental example of the mean curvature equation for graphs. We provide an application of the uniqueness theorem to the mean curvature flow of entire graphs. The last sections 4 and 5 are devoted to the proofs of the main theorems.

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†A function $m : I\!R^+ \to I\!R^+$ is said to be a modulus of continuity if $m(0_+) := \lim_{s\to 0_+} m(s) = 0$ and $m(t+s) \leq m(t) + m(s)$ for any $s, t \in I\!R^+$.

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2 Statement of the results and examples

Before stating the problem and our results, we introduce some notations. In the sequel, $\mathcal{M}_{N,M}$ is the set of $N \times M$–matrices and $\mathcal{S}_N$ (respectively $\mathcal{S}_N^+$) denotes the set of the symmetric (respectively symmetric nonnegative) matrices. For every $A \in \mathcal{M}_{N,M}$, A^T denotes the transpose of A. Finally, we introduce the space $\mathcal{C}_{\text{poly}}$ of locally bounded, possibly discontinuous, functions on $I\!R^N \times [0,T]$ which have a polynomial growth with respect to the space variable. More precisely, $u \in \mathcal{C}_{\text{poly}}$ if there exists a constant $k > 0$ such that

$$\frac{u(x,t)}{1+|x|^k} \underset{|x|\to+\infty}{\longrightarrow} 0, \text{ uniformly with respect to } t \in [0,T].$$

We will use the following assumptions for equation (2) :

(H1) There exists a continuous function $\sigma : I\!R^N \times [0,T] \times I\!R^N \to \mathcal{M}_{N,M}$ and some constant $\tilde{C}$ such that

$$b(x,t,p) = \sigma(x,t,p)\sigma(x,t,p)^T,$$

$$\|\sigma(x,t,p) - \sigma(y,t,p)\| \leq \tilde{C}|x-y|,$$

$$\|\sigma(x,t,p) - \sigma(x,t,q)\| \leq \tilde{C}\frac{|p-q|}{1+|p|+|q|}.$$

(H2) The function H is continuous on $I\!R^N \times [0,T] \times I\!R \times I\!R^N$, $u \mapsto H(x,t,u,p)$ is nondecreasing for every (x,t,p) and there exists a modulus of continuity $\tilde{m}$ such that, for every $x,y,p,q \in I\!R^N$, $u \in I\!R$ and $t \in [0,+\infty)$,

$$|H(x,t,u,p) - H(y,t,u,p)| \leq \tilde{m}\left((1+|p|)|x-y|\right),$$

and

$$|H(x,t,u,p) - H(x,t,u,q)| \leq \tilde{m}\left((1+|x|)|p-q|\right).$$

Finally we say that a function $\omega : I\!R^N \to I\!R$ satisfies the assumption **(H3-ω)**(ν) if there exists a modulus of continuity m such that, for every $x,y \in I\!R^N$,

$$|\omega(x) - \omega(y)| \leq m\left((1+|x|+|y|)^\nu |x-y|\right).$$

Our main result is the following

Theorem 2.1 *Assume that* ***(H1)*** *and* ***(H2)*** *hold and let* $u \in \mathcal{C}_{\rm poly}$ *(respectively* $v \in \mathcal{C}_{\rm poly}$*) be a upper-semicontinuous (respectively lower-semicontinuous) viscosity subsolution (respectively viscosity supersolution) of (2). If*

$$u(x,0) \le u_0(x) \le v(x,0) \quad \text{in } I\!\!R^N ,$$

where u_0 *is a function which satisfies* ***(H3-***u_0***)***(ν) *with* $0 \le \nu < (1+\sqrt{5})/2$*, then*

$$u \le v \qquad \text{in } I\!\!R^N \times [0,T].$$

The proof is postponed to Section 4. An immediate consequence of Theorem 2.1 is the

Corollary 2.1 *Assume that* ***(H1)*** *and* ***(H2)*** *hold and let* u_0 *be a function which satisfies* ***(H3-***u_0***)***(ν)*, with* $0 \le \nu < (1+\sqrt{5})/2$*. In* $\mathcal{C}_{\rm poly}$*, there exists a unique continuous viscosity solution* u *of (2). Moreover, there exists* $C > 0$ *such that*

$$|u(x,t)| \le C(1+|x|^{\nu+1}).$$

The existence of a solution is a consequence of the comparison result by using Perron's method. A priori, this existence results takes place in the class of functions with polynomial growth but we prove that the solution inherits the growth of the initial datum. The proof will be given in Section 5.

Remark 2.1 : Concerning the existence of classical solutions to these equations, we refer to Chou and Kwong [8] who provided $W^{1,\infty}$ local bounds for the solutions of a large class of quasilinear equations without growth restriction on the initial data. The existence of a smooth solution to the mean curvature equation for graphs, also in the case when there is no restriction on the initial data, was first shown by Ecker and Huisken [12] and Evans and Spruck [14] (see also [3]).

Before giving examples of equations to which these results apply, let us make some comments about the assumption **(H1)** and **(H3-**ω**)**(ν), **(H2)** being a natural and classical assumption. In **(H1)**, the two first conditions are classical ; note that the first one implies that the equation is degenerate parabolic. Of course, the third one is the most interesting since it concerns the behavior of b in the gradient variable and we recall that it is a key difficulty here. We have chosen this type of assumption for σ in p because this is the type of dependence we have for (1). We have decided not to consider in this paper different choices of p–dependence in order to keep the length of this paper reasonable. They would lead to results with, in particular, different limitation on the growth of the initial data.

It is anyway worth pointing out that, with such type of assumptions on σ, there is no hope to prove a better result than a comparison principle in $\mathcal{C}_{\rm poly}$ or for solutions with exponential growth: indeed the heat equation satisfies **(H1)**. In order to go further, one has to take in account some degeneracy of the equation (typically $\sigma(p)p \to 0$ as $|p| \to +\infty$) but our proof does not see at all this kind of property.

Notice that the assumptions on σ yield the existence of a positive constant C such that, for every $(x,t,p) \in I\!R^N \times [0,T] \times I\!R^N$,

$$\|\sigma(x,t,p)\| \le C(1+|x|)\,, \tag{4}$$

a key fact in the proof.

A more readable version of assumption **(H3-ω)**(ν) on the initial data of the equation is when ω is locally Lipschitz continuous, then **(H3-ω)** holds if

$$|D\omega(x)| \le C\,(1+|x|^\nu) \quad \text{for almost every } x \in I\!R^N.$$

We end this section by a list of equations for which the previous results apply.

1. Equations of the form

$$\frac{\partial u}{\partial t} - a(x,t)\frac{\Delta u}{(1+|Du|^2)^\alpha} + H(x,t,u,Du) = 0.$$

The above results apply when $\alpha > 0$, H satisfies **(H2)** and $a \in C(I\!R^N \times [0,+\infty))$ is a bounded nonnegative function such that $\sqrt{a}$ is Lipschitz continuous in x, uniformly with respect to t. Since it is not obvious, we check the third statement of **(H1)**. Without loss of generality, we can suppose $a \equiv 1$, the dimension of the space $N=1$ and $q \ge p \ge 0$. We have

$$\sigma(p) - \sigma(q) = \frac{1}{(1+p^2)^{\alpha/2}}\left(1 - \left(1 + \frac{p^2-q^2}{1+q^2}\right)^{\alpha/2}\right).$$

From the inequality $1-(1+x)^{\alpha/2} \le -x$ for $-1 \le x \le 0$, we get

$$\sigma(p) - \sigma(q) \le \frac{1}{(1+p^2)^{\alpha/2}}\frac{(q-p)(p+q)}{1+q^2} \le \frac{2q}{1+q^2}(p-q).$$

Since we assumed $0 \le p \le q$, we can find a constant C independent of p,q such that $2q/(1+q^2) \le C/(1+p+q)$. It gives the conclusion.

2. Non geometric curvature flow:

$$\frac{\partial u}{\partial t} - \mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} = 0.$$

3. Consider

$$\frac{\partial u}{\partial t} - \mathrm{Tr}[A(x,t)\left(I - \frac{Du \otimes Du}{1+|Du|^2}\right)A(x,t)D^2u] + H(x,t,u,Du) = 0,$$

If A is a bounded continuous function from $I\!R^N \times [0,+\infty)$ into $\mathcal{M}_N$, Lipschitz in x (uniformly in t) and H satisfies **(H2)**, then our results apply. In particular,

$$\frac{\partial u}{\partial t} - a(x,t)\sqrt{1+|Du|^2}\left(\mathrm{div}\frac{Du}{\sqrt{1+|Du|^2}} + c(x,t)\right) = 0$$

fulfills the assumptions as soon as the continuous functions a, c are bounded, a is nonnegative and $\sqrt{a}, c$ are Lipschitz continuous in x uniformly in t. If $a \equiv 1$ and c is constant, then we recognize the geometric eikonal equation. The mean curvature equation for graphs ($a \equiv 1$ and $c \equiv 0$) is studied in great details in the following section.

4.

$$\frac{\partial u}{\partial t} - \sup_{\alpha\in\mathcal{A}} \mathrm{Tr}[b_\alpha(x,t,Du)D^2u] + \sup_{\beta\in\mathcal{B}} H_\beta(x,t,u,Du) = 0,$$

when **(H1)** holds for b_α and **(H2)** holds for H_β with constants independent of α, β.

5. With minor adaptations in the proof of the comparison result we can deal with

$$\frac{\partial u}{\partial t} - f\left(\mathrm{Tr}[b(x,t,Du)D^2u]\right) + H(x,t,u,Du) = 0,$$

under **(H1)** and **(H2)** for a nondecreasing nonnegative function f which satisfies, for every $z, z' \in I\!R$,

$$f(z) - f(z') \leq C|z'-z|^\alpha,$$

with $\alpha \in [0,1]$.

3 Application to the mean curvature equation

Our main motivation is to prove some uniqueness results for the mean curvature equation for graphs. In this section, we recall some facts about this equation and show that it enters the framework of Theorem 2.1. We then apply these results to the mean curvature flow for entire graphs in $I\!R^{N+1}$.

We can write the mean curvature equation in different forms. To follow the notations of the previous section, we will write it as

$$\frac{\partial u}{\partial t} - \mathrm{Tr}\left[b_c(Du)\ D^2u\right] = 0, \tag{5}$$

where, for every $p \in I\!R^N$,

$$b_c(p) = I - \frac{p\otimes p}{1+|p|^2}\ . \tag{6}$$

The same equation is often written

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2uDu, Du\rangle}{1+|Du|^2} = 0 \quad \text{in } I\!R^N \times (0,+\infty), \tag{7}$$

or in divergence form

$$\frac{\partial u}{\partial t} - \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0. \tag{8}$$

The operator $b_{\rm c}$ defined by (6) maps $I\!R^N$ into $\mathcal{S}_N^+$. For every $p \in I\!R^N$, we can define the positive symmetric square root $b_{\rm c}^{1/2}(p)$ of $b_{\rm c}(p) \in \mathcal{S}_N^+$. The following lemma provides some properties of $b_{\rm c}^{1/2}$.

Lemma 3.1 *For every $p \in I\!R^N$,*

$$b_{\rm c}^{1/2}(p) = I - \frac{1}{\sqrt{1+|p|^2}(1+\sqrt{1+|p|^2})} p \otimes p.$$

Moreover, $b_{\rm c}^{1/2}$ is bounded and Lipschitz continuous in $I\!R^N$; there exists a positive constant C such that, for every $p, q \in I\!R^N$,

$$\|b_{\rm c}^{1/2}(p) - b_{\rm c}^{1/2}(q)\| \le \frac{C|p-q|}{1+|p|+|q|}. \tag{9}$$

Proof of Lemma 3.1. We first compute $b_{\rm c}^{1/2}$. For every $q \in (\mathrm{Span}\, p)^\perp$, $b_{\rm c}(p)q = q$ and $b_{\rm c}(p)p = p/(1+|p|^2)$. It follows that we can look for $b_{\rm c}^{1/2}$ in the form $b_{\rm c}^{1/2}(p) = I - f(p) p \otimes p$. Then, an easy calculation gives the result. Looking at the formula, it is clear that $b_{\rm c}^{1/2}$ is continuous and bounded.

To check (9), we write

$$\begin{aligned} \|b_{\rm c}^{1/2}(p) - b_{\rm c}^{1/2}(q)\| &= \|f(q) q\otimes q - f(p) p \otimes p\| \\ &= \|q \otimes (q-p) f(q) + (q-p)\otimes p f(q) + p \otimes p (f(q)-f(p))\| \\ &\le (|q| f(q) + |p| f(q))|p-q| + |p|^2 |f(p)-f(q)|. \end{aligned}$$

Without loss of generality, we can take $|q| \ge |p|$. On the one hand,

$$\begin{aligned} |q|f(q) + |p|f(q) &\le 2|q|f(q) \le \frac{2|q|}{\sqrt{1+|q|^2}(1+\sqrt{1+|q|^2})} \le \frac{2}{1+\sqrt{1+|q|^2}} \\ &\le \frac{2}{1+|q|} \le \frac{2}{1+\frac{1}{2}(|p|+|q|)} \le \frac{4}{1+|p|+|q|}. \end{aligned} \tag{10}$$

On the other hand, by setting $g(p) = \sqrt{1+|p|^2}$,

$$\begin{aligned} |p|^2 |f(q) - f(p)| &\le |p|^2 f(p) f(q)\, |g(p)(1+g(p)) - g(q)(1+g(q))| \\ &\le |p|^2 f(p) f(q)\, |1+g(p)+g(q)|\, |g(p)-g(q)|\,. \end{aligned}$$

But a straightforward calculation shows that $|g(p)-g(q)| \leq |p-q|$ and that the application

$$(p,q) \mapsto |p|^2 f(p) f(q) |1+g(p)+g(q)| (1+|p|+|q|)$$

is continuous and bounded on the set $\{(p,q) \in I\!R^{2N} : |p| \leq |q|\}$ thus there exists a constant $C_1 > 0$ such that for every $p, q \in I\!R^N$, $|p| \leq |q|$, we have

$$|p|^2 f(p) f(q) \, |1+g(p)+g(q)| \, |g(p)-g(q)| \leq \frac{C_1 |p-q|}{1+|p|+|q|}. \tag{11}$$

Combining (10) and (11), we obtain the result with $C = 4 + C_1$. □

Thanks to the previous lemma and the existence of a smooth solution to (7) (cf. [12], [14] and [3]) we get

Theorem 3.1 *Assume that* $u_0 \in C(I\!R^N)$ *satisfies* ***(H3-**u_0**)**(ν), *with* $0 \leq \nu < (1+\sqrt{5})/2$. *Then (7) (or equivalently (5) or (8)) has a unique solution* $u \in C(I\!R^N \times [0,+\infty)) \cap C^\infty(I\!R^N \times (0,+\infty))$ *in* $\mathcal{C}_{\rm poly}$.

We turn to a geometrical application to the so-called level-set approach to the generalized evolution of hypersurfaces by their mean curvature. This method, introduced for numerical computations by Osher and Sethian [19], was developed theoretically by Evans and Spruck [13] and Chen, Giga and Goto [7].

Let us recall briefly the level-set approach in the case of the mean curvature motion of entire graphs. We consider the graph of the initial datum $u_0 \in C(I\!R)$ of (7) as an hypersurface $\Gamma_0 = \{(x,y) \in I\!R^N \times I\!R : u_0(x) = y\}$ of $I\!R^{N+1}$. Define $\Omega_0 = \{(x,y) \in I\!R^{N+1} : u_0(x) < y\}$. We take a uniformly continuous function $v_0 : I\!R^{N+1} \to I\!R$ such that

$$\Gamma_0 = \{(x,y) \in I\!R^{N+1} : v_0(x,y) = 0\} \quad \text{and} \quad \Omega_0 = \{(x,y) \in I\!R^{N+1} : v_0(x,y) > 0\} \tag{12}$$

(choose the signed-distance to Γ_0 for instance). Next, we consider a function $v : I\!R^{N+1} \times (0,+\infty) \to I\!R$ such that $v(x,u(x,t),t) = 0$ where u is a solution of (7). Formally, v has to satisfy the well-known geometrical mean curvature equation

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + \dfrac{\langle D^2 v Dv, Dv\rangle}{|Dv|^2} & \text{in } I\!R^{N+1} \times (0,T), \\ v(\cdot,\cdot,0) = v_0 \quad \text{in } I\!R^{N+1}. \end{cases}$$

This equation admits a unique viscosity solution $v \in UC(I\!R^{N+1} \times [0,+\infty))$ for every initial datum $v_0 \in UC(I\!R^{N+1})$, where UC denotes the uniformly continuous functions. Moreover, the level-set approach works: it means that we can define, for every $t \in [0,T]$,

$$\Gamma_t = \{(x,y) \in I\!R^{N+1} : v(x,y,t) = 0\} \quad \text{and} \quad \Omega_t = \{(x,y) \in I\!R^{N+1} : v(x,y,t) > 0\};$$

the sets $(\Gamma_t)_t$ and $(\Omega_t)_t$ depend only on the initial sets Γ_0 and Ω_0 but not on the choice of v_0. The family $(\Gamma_t)_t$ is called *the generalized evolution by mean curvature of the graph* Γ_0 and Γ_t is called the *front*. A natural issue is the connection between this generalized evolution and the classical motion by mean curvature of the graph of u_0. Note that Γ_t is defined as the 0-level-set of a uniformly continuous function; it may be very irregular in general and can even develop an interior in $I\!R^{N+1}$. In our context, we have

Theorem 3.2 *If $u_0 \in C(I\!R^N)$ satisfies **(H3-**u_0**)**(ν), with $0 \le \nu < (1+\sqrt{5})/2$, then, for every $t \in [0,T]$, the set Γ_t is a entire smooth graph, namely*

$$\Gamma_t = \{(x,y) \in I\!R^{N+1} : y = u(x,t)\},$$

where u is the unique smooth solution of (7) with initial datum u_0. Moreover, the evolution of Γ_t agrees with the classical motion by mean curvature in the sense of the differential geometry.

Proof of Theorem 3.2. From [3], we know that, if we start with an hypersurface Γ_0 which is an entire continuous graph in $I\!R^N \times I\!R$, then, for every $t \in [0,T]$,

$$\Gamma_t = \{(x,y) \in I\!R^{N+1} : u^-(x,t) \le y \le u^+(x,t)\},$$

where u^- and u^+ are respectively the minimal and the maximal (discontinuous) viscosity solution of (7). In the special case of the mean curvature equation, we proved that the boundary of the front Γ_t is smooth. It follows that u^- and u^+ are smooth. At this step, we aim at applying Theorem 3.1 to show that $u^- = u^+$. To do it, we need to know that $u^-, u^+ \in \mathcal{C}_{\rm poly}$. Note that Corollary 2.1 is not sufficient because we suppose, a priori, that the solution has a polynomial growth. To overcome this difficulty, we invoke a L^∞ local bound for (7) (see [3]): there exists a constant C such that, for every $(x,t) \in I\!R^N \times [0,T]$,

$$|u(x,t)| \le \max\{|u_0(y)| + \sqrt{2Ct} : y \in \bar{B}(x, \sqrt{2Ct})\}.$$

From **(H3-**u_0**)**(ν), there exists a positive constant $\tilde{C}$ such that $|u_0(y)| \le \tilde{C}(1+|y|^{1+\nu})$. It follows

$$|u(x,t)| \le \tilde{C}(1 + (|x| + \sqrt{2Ct})^{1+\nu}) + \sqrt{2CT},$$

which proves the claim.

It implies $u^-, u^+ \in \mathcal{C}_{\rm poly}$ and from Theorem 3.1, we obtain $u^- = u^+ := u$. Finally, $\Gamma_t = {\rm Graph}(u(\cdot,t))$ is a smooth submanifold of $I\!R^{N+1}$ (in particular, Γ_t never fattens). In this case, the generalized evolution fits in with the classical evolution by mean curvature (see Evans and Spruck [13] and [14] for the agreement with an alternative generalized motion). □

We refer to [3] and [5] for some more general geometrical motions. In these works, we associate a geometrical motion to some quasilinear equations of the type (2) for which the above techniques apply.

4 Proof of Theorem 2.1

1. We argue by contradiction assuming that there exists $(\tilde{x},\tilde{t}) \in I\!R^N \times [0,T]$ such that

$$u(\tilde{x},\tilde{t}) > v(\tilde{x},\tilde{t}). \tag{13}$$

For $\varepsilon, \alpha > 0, p \geq k > \max\{2, \nu+1\}$ to be chosen later on, we introduce the functions defined by : for every $x', y' \in I\!R^N$,

$$K(x') := 1 + |x'|^k, \quad \varphi(y') := \frac{|y'|^p}{\varepsilon^p} \quad \text{and} \quad \Psi(x',y') := K(x')(\varphi(y') + \alpha).$$

Then we consider the test-function given by : for every $x, y \in I\!R^N$,

$$\Phi(x,y,t) := \mathrm{e}^{Lt}\, \Psi(x+y, x-y) + \eta t,$$

where $L, \eta > 0$.

2. The first step of the proof is the

Lemma 4.1 *Under the assumptions of Theorem 2.1, there exists a constant $C > 0$ such that, for any $x \in I\!R^N$ and $t \in [0,T]$,*

$$u(x,t) \leq C(1+|x|^{\nu+1}) \quad \textit{and} \quad v(x,t) \geq -C(1+|x|^{\nu+1})\,. \tag{14}$$

Moreover, if $p \geq k > \nu + 1$, the

$$\sup_{(x,y,t)\in (I\!R^N)^2\times[0,T]} \big\{u(x,t) - v(y,t) - \Phi(x,y,t)\big\}$$

is finite and is achieved at a point $(\bar{x},\bar{y},\bar{t})$. Finally, we can choose the parameters η, α and ε small enough in order to have a positive supremum and $|\bar{x}-\bar{y}| \leq 1$.

We postpone the proof of this lemma to the end of the section. From now on, we suppose that the supremum is positive and that $|\bar{x}-\bar{y}| \leq 1$.

3. The idea of the proof is the following : the seven next steps are devoted to show that we can fix the parameters in Φ in order to force the maximum to be achieved at time $t = 0$. The last step deals with the case $t = 0$. We prove that the particular form of the test-function and the assumption about the modulus of continuity of the initial data lead to a contradiction with (13) which will be the end of the proof.

4. We first consider the case when the supremum is achieved at a point such that $\bar{t} > 0$.

By applying the fundamental result of the Users' guide to viscosity solutions ([9, Theorem 8.3]), we know that, for every $\rho > 0$, there exist $a_1, a_2 \in I\!R$ and $X, Y \in \mathcal{S}_N$ such that

$$(a_1, D_x\Phi(\bar{x},\bar{y},\bar{t}), X) \in \bar{\mathcal{P}}^{2,+}(u)(\bar{x},\bar{t})\,,$$

$$(a_2, -D_y\Phi(\bar x,\bar y,\bar t), Y) \in \bar{\mathcal{P}}^{2,-}(v)(\bar y,\bar t)\ ,$$
$$a_1 - a_2 = \frac{\partial \Phi}{\partial t}(\bar x,\bar y,\bar t)\ ,$$

and

$$\left(\begin{array}{cc} X & 0 \\ 0 & -Y \end{array}\right) \leq A + \rho A^2 \quad \text{where} \quad A = D^2\Phi(\bar x,\bar y,\bar t)\ . \tag{15}$$

Therefore, since u and v are respectively sub- and supersolution of (2), we have

$$a_1 - \mathrm{Tr}\,[b(\bar x,\bar t, D_x\Phi(\bar x,\bar y,\bar t))X] + H(\bar x,\bar t,u(\bar x,\bar t), D_x\Phi(\bar x,\bar y,\bar t)) \leq 0 \tag{16}$$

and

$$a_2 - \mathrm{Tr}\,[b(\bar y,\bar t,-D_y\Phi(\bar x,\bar y,\bar t))Y] + H(\bar y,\bar t,v(\bar y,\bar t),-D_y\Phi(\bar x,\bar y,\bar t)) \geq 0. \tag{17}$$

By subtracting (17) from (16), we obtain

$$\begin{aligned}\eta + L\Phi(\bar x,\bar y,\bar t) \ \leq\ & \mathrm{Tr}\,[b(\bar x,\bar t, D_x\Phi(\bar x,\bar y,\bar t))X] - \mathrm{Tr}\,[b(\bar y,\bar t,-D_y\Phi(\bar x,\bar y,\bar t))Y] \\ & + H(\bar y,\bar t,v(\bar y,\bar t),-D_y\Phi(\bar x,\bar y,\bar t)) - H(\bar x,\bar t,u(\bar x,\bar t), D_x\Phi(\bar x,\bar y,\bar t)).\end{aligned} \tag{18}$$

In order to show that the maximum cannot be achieved for $\bar t > 0$, we have to prove that this inequality cannot hold for a suitable choice of the parameters and, to do so, we have to estimate the right-hand side of this inequality.

5. For the sake of notational simplicity, we write

$$\sigma_x = \sigma(\bar x,\bar t, D_x\Phi(\bar x,\bar y,\bar t)) \quad \text{and} \quad \sigma_y = \sigma(\bar y,\bar t,-D_y\Phi(\bar x,\bar y,\bar t))$$

and omit in the following computations the dependence on $(\bar x,\bar y,\bar t)$ when there is no ambiguity. For any orthonormal basis $(e_i)_{1\leq i\leq N=1}$ of $I\!R^N$, we have

$$\begin{aligned}\mathrm{Tr}\,[b(\bar x,\bar t, D_x\Phi(\bar x,\bar y,\bar t))X - b(\bar y,\bar t,-D_y\Phi(\bar x,\bar y,\bar t))Y] &= \mathrm{Tr}\,\left[\sigma_x^T X\sigma_x - \sigma_y^T Y\sigma_y\right] \\ &= \sum_{i=1}^N \langle X\sigma_x e_i, \sigma_x e_i\rangle - \langle Y\sigma_y e_i,\sigma_y e_i\rangle\ .\end{aligned} \tag{19}$$

On an other hand, we can write (15) as, for all $\zeta,\xi \in I\!R^N$,

$$\begin{aligned}\langle X\zeta,\zeta\rangle - \langle Y\xi,\xi\rangle \leq e^{L\bar t}\langle D^2_{xx}\Psi(\zeta+\xi),\zeta+\xi\rangle + 2e^{L\bar t}\langle D^2_{xy}\Psi(\zeta+\xi),\zeta-\xi\rangle \\ + e^{L\bar t}\langle D^2_{yy}\Psi(\zeta-\xi),\zeta-\xi\rangle + \rho\langle A^2(\zeta,\xi),(\zeta,\xi)\rangle.\end{aligned}$$

Using this last inequality with $\zeta = \sigma_x e_i$ and $\xi = \sigma_y e_i$ in (19) and letting ρ go to 0, (18) becomes

$$\eta + L\Phi \leq \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4, \tag{20}$$

where

$$\begin{cases} \mathcal{A}_1 = e^{L\bar{t}} \|D^2_{xx}\Psi\| \, \|\sigma_x + \sigma_y\|^2, \\ \mathcal{A}_2 = 2e^{L\bar{t}} \|D^2_{xy}\Psi\| \, \|\sigma_x + \sigma_y\| \, \|\sigma_x - \sigma_y\|, \\ \mathcal{A}_3 = e^{L\bar{t}} \|D^2_{yy}\Psi\| \, \|\sigma_x - \sigma_y\|^2, \\ \mathcal{A}_4 = H(\bar{y}, \bar{t}, v(\bar{y}, \bar{t}), -D_y\Phi) - H(\bar{x}, \bar{t}, u(\bar{x}, \bar{t}), D_x\Phi). \end{cases}$$

In the fourth next steps, we estimate the $\mathcal{A}_i, 1 \leq i \leq 4$. These estimates are heavy to obtain but the basic idea is easy: we want to prove that, for a suitable choice of parameters, each $|\mathcal{A}_i|$ is bounded by $\bar{C}\Phi + \beta$, where $\bar{C}$ is a fixed constant and β is sufficiently small. It will lead to a contradiction with (20) if we take L large enough.

In all the estimates below, C will denote a constant which may vary from line to line, may depend on k, p but not on ε, α and L.

6. Estimate of $\mathcal{A}_1$.
We have

$$\|D_{xx}\Psi\| \leq k(k-1)|\bar{x} + \bar{y}|^{k-2}(\varphi + \alpha).$$

From (4), we get

$$\mathcal{A}_1 \;\leq\; 2e^{L\bar{t}} C^2 k(k-1)|\bar{x} + \bar{y}|^{k-2}((1 + |\bar{x}|)^2 + (1 + |\bar{y}|)^2)(\varphi + \alpha).$$

But, since $|\bar{x} - \bar{y}| \leq 1$, we can compare both $|\bar{x}|$ and $|\bar{y}|$ with $|\bar{x} + \bar{y}|$. It follows that there exists a constant $C = C(k)$ such that

$$\mathcal{A}_1 \leq C e^{L\bar{t}} K(\varphi + \alpha). \tag{21}$$

7. Estimate of $\mathcal{A}_2$.
An explicit computation gives $D^2_{xy}\Psi = DK \otimes D\varphi$. We may assume without loss of generality that $\bar{x} \neq \bar{y}$ or equivalently $D\varphi \neq 0$ since otherwise $\mathcal{A}_2$ would be 0 and causes no problem.

From inequalities (4) and $|\bar{x} - \bar{y}| \leq 1$, we then obtain

$$\mathcal{A}_2 \leq 2Ce^{L\bar{t}} |DK||D\varphi|(3 + |\bar{x} + \bar{y}|)\|\sigma_x - \sigma_y\|. \tag{22}$$

Using **(H1)**, we have

$$\begin{aligned} \|\sigma_x - \sigma_y\| \;&\leq\; \|\sigma(\bar{x}, \bar{t}, D_x\Phi) - \sigma(\bar{y}, \bar{t}, D_x\Phi)\| + \|\sigma(\bar{y}, \bar{t}, D_x\Phi) - \sigma(\bar{y}, \bar{t}, -D_y\Phi)\| \\ &\leq\; C|\bar{x} - \bar{y}| + C\frac{|D_x\Phi + D_y\Phi|}{1 + |D_x\Phi| + |D_y\Phi|}. \end{aligned}$$

But $D_x\Phi = e^{L\bar{t}}(DK(\varphi+\alpha)+KD\varphi)$ and $D_y\Phi = e^{L\bar{t}}(DK(\varphi+\alpha)-KD\varphi)$. We obtain

$$\begin{aligned}\|\sigma_x-\sigma_y\| &\le C|\bar{x}-\bar{y}| + C\frac{2e^{L\bar{t}}|DK|(\varphi+\alpha)}{1+\frac{1}{2}(|D_x\Phi+D_y\Phi|+|D_x\Phi-D_y\Phi|)}\\ &\le C|\bar{x}-\bar{y}| + 2C\frac{|DK|(\varphi+\alpha)}{K|D\varphi|}. \qquad (23)\end{aligned}$$

Combining this last inequality with (22) and using that

$$|DK| = k|\bar{x}+\bar{y}|^{k-1} \quad \text{and} \quad |D\varphi| = p\,\frac{|\bar{x}-\bar{y}|^{p-1}}{\varepsilon^p},$$

show that there exists a positive constant $C=C(k,p)$ such that

$$\mathcal{A}_2 \le Ce^{L\bar{t}}K(\varphi+\alpha). \qquad (24)$$

8. Estimate of $\mathcal{A}_3$.
This step is the most technical one. First, the computation of $D^2_{yy}\Psi$ gives

$$\mathcal{A}_3 \le e^{L\bar{t}}K\|D^2\varphi\|\;\|\sigma_x-\sigma_y\|^2$$

and we have

$$\begin{aligned}\|\sigma_x-\sigma_y\|^2 \;=\;& \|\sigma(\bar{x},\bar{t},D_x\Phi)-\sigma(\bar{y},\bar{t},D_x\Phi)\|^2\\ &+2\,\|\sigma(\bar{x},\bar{t},D_x\Phi)-\sigma(\bar{y},\bar{t},D_x\Phi)\|\,\|\sigma(\bar{y},\bar{t},D_x\Phi)-\sigma(\bar{y},\bar{t},-D_y\Phi)\|\\ &+\|\sigma(\bar{y},\bar{t},D_x\Phi)-\sigma(\bar{y},\bar{t},-D_y\Phi)\|^2.\end{aligned}$$

It follows

$$\begin{aligned}|\mathcal{A}_3| \;\le\;& 2K\,\mathrm{e}^{L\bar{t}}\,\|\sigma(\bar{x},\bar{t},D_x\Phi)-\sigma(\bar{y},\bar{t},D_x\Phi)\|^2\,\|D^2\varphi\|\\ &+2K\,\mathrm{e}^{L\bar{t}}\,\|\sigma(\bar{y},\bar{t},D_x\Phi)-\sigma(\bar{y},\bar{t},-D_y\Phi)\|^2\,\|D^2\varphi\|. \qquad (25)\end{aligned}$$

We estimate separately the two terms which appear in the right-hand side (25). For the first one, using **(H1)**, we obtain

$$2K\,\mathrm{e}^{L\bar{t}}\,\|\sigma(\bar{x},\bar{t},D_x\Phi)-\sigma(\bar{y},\bar{t},D_x\Phi)\|^2\,\|D^2\varphi\| \le 2C^2p(p-1)\,\mathrm{e}^{L\bar{t}}K\varphi \qquad (26)$$

because $|\bar{x}-\bar{y}|^2\|D^2\varphi\| \le p(p-1)\varphi$.

The estimate of the second term is the hardest one. Again we may assume without loss of generality that $\bar x \neq \bar y$ or equivalently $D\varphi \neq 0$; we have

$$
\begin{aligned}
\text{second term } &= 2K\,\mathrm{e}^{L\bar t}\,\|\sigma(\bar y,\bar t,D_x\Phi)-\sigma(\bar y,\bar t,-D_y\Phi)\|^2\,\|D^2\varphi\| \\
&\leq 2CK\mathrm{e}^{L\bar t}\ \frac{|\bar x-\bar y|^{p-2}}{\varepsilon^p}\ \left(\frac{C|D_x\Phi+D_y\Phi|}{1+|D_x\Phi|+|D_y\Phi|}\right)^2 \\
&\leq 2CK\mathrm{e}^{L\bar t}\ \frac{|\bar x-\bar y|^{p-2}}{\varepsilon^p}\ \left(\frac{|D_x\Phi+D_y\Phi|}{1+\frac12\left(|D_x\Phi+D_y\Phi|+|D_x\Phi-D_y\Phi|\right)}\right)^2 \\
&\leq 2CK\mathrm{e}^{L\bar t}\ \frac{|\bar x-\bar y|^{p-2}}{\varepsilon^p}\ \left(\frac{2\mathrm{e}^{L\bar t}(\varphi+\alpha)|DK|}{1+\mathrm{e}^{L\bar t}(\varphi+\alpha)|DK|+\mathrm{e}^{L\bar t}K|D\varphi|}\right)^2 \\
&\leq 2CK\mathrm{e}^{L\bar t}\ \frac{|\bar x-\bar y|^{p-2}}{\varepsilon^p}\ \frac{\left(\mathrm{e}^{L\bar t}(\varphi+\alpha)|DK|\right)^2}{\left(\mathrm{e}^{L\bar t}(\varphi+\alpha)|DK|\right)^{2\lambda}\left(\mathrm{e}^{L\bar t}K|D\varphi|\right)^{2\mu}}
\end{aligned}
$$

for every positive λ,μ such that $\lambda+\mu\leq 1$.

We now distinguish two cases.

— 1^{st} case. If $\varphi \geq \alpha$, then taking $\lambda = 0$ and $\mu = 1$, we get

$$
\begin{aligned}
\text{second term } &\leq 2CK\mathrm{e}^{L\bar t}\ \frac{|x-y|^{p-2}}{\varepsilon^p}\ \frac{\mathrm{e}^{2L\bar t}\varphi^2|DK|^2}{\mathrm{e}^{2L\bar t}K^2|D\varphi|^2} \\
&\leq 2CK\mathrm{e}^{L\bar t}\varphi\left(\frac{|DK|}{K}\right)^2 \\
&\leq 2CK\mathrm{e}^{L\bar t}\varphi. \qquad (27)
\end{aligned}
$$

— 2^{nd} case. Otherwise, $\alpha \geq \varphi$. Then

$$
\begin{aligned}
\text{second term } &\leq 2CK\mathrm{e}^{L\bar t}\frac{|\bar x-\bar y|^{p-2}}{\varepsilon^p}\ \frac{\left(\mathrm{e}^{L\bar t}\alpha|DK|\right)^2}{\left(\mathrm{e}^{L\bar t}\alpha|DK|\right)^{2\lambda}\left(\mathrm{e}^{L\bar t}K\frac{p|\bar x-\bar y|^{p-1}}{\varepsilon^p}\right)^{2\mu}} \\
&\leq 2CK^{1-2\mu}\mathrm{e}^{L\bar t(3-2\lambda-2\mu)}|\bar x-\bar y|^{p-2-2\mu(p-1)}\ \varepsilon^{-p+2\mu p}\ \ (\alpha|DK|)^{2-2\lambda}\ .
\end{aligned}
$$

We choose $\mu = \dfrac{p-2}{2(p-1)}$ and this yields

$$\text{second term} \leq 2C\mathrm{e}^{3L\bar t}\varepsilon^{-\frac{p}{p-1}}\alpha^{2-2\lambda}K^{\frac{1}{p-1}}|DK|^{2-2\lambda}.$$

From now on, we take

$$p > k+1 \qquad (28)$$

and we choose $\lambda = \dfrac12 - \dfrac{\theta}{2}$ with

$$\theta = \frac{p-k-1}{(p-1)(k-1)}. \qquad (29)$$

Using that $|DK| \leq kK^{\frac{k-1}{k}}$, we obtain

$$\begin{aligned}\text{second term} &\leq 2C\mathrm{e}^{3L\bar{t}}\varepsilon^{-\frac{p}{p-1}}\alpha^{2-2\lambda}K^{\frac{1}{p-1}+\frac{(2-2\lambda)(k-1)}{k}} \\ &\leq 2C\mathrm{e}^{3L\bar{t}}\varepsilon^{-\frac{p}{p-1}}\alpha\alpha^{\theta}K\,. \end{aligned} \tag{30}$$

Combining (27) and (30), we obtain

$$\text{second term} \leq \max\left\{2C\mathrm{e}^{L\bar{t}}K\varphi,\ 2C\mathrm{e}^{3L\bar{t}}\varepsilon^{-p/(p-1)}\ \alpha^{\theta}\ \alpha K\right\}, \tag{31}$$

and from (26) and (31), we get

$$\begin{aligned}|\mathcal{A}_3| &\leq C\mathrm{e}^{L\bar{t}}K\varphi + CK\mathrm{e}^{L\bar{t}}(\varphi+\alpha) \\ &\quad +\max\left\{2C\mathrm{e}^{L\bar{t}}K\varphi,\ 2C\mathrm{e}^{3L\bar{t}}\ \varepsilon^{-p/(p-1)}\ \alpha^{\theta}\ \alpha K\right\} \\ &\leq C\max\left\{\mathrm{e}^{L\bar{t}}K(\varphi+\alpha),\ \mathrm{e}^{3L\bar{t}}\ \varepsilon^{-p/(p-1)}\ \alpha^{\theta}\ \alpha K\right\}, \end{aligned} \tag{32}$$

where C is a positive constant independent of ε, α, L and η.

9. Estimate of $\mathcal{A}_4$.
Since we have

$$0 < \Phi(\bar{x},\bar{y},\bar{t}) \leq u(\bar{x},\bar{t}) - v(\bar{y},\bar{t})$$

and since $u \mapsto H(\cdot,\cdot,u,\cdot)$ is nondecreasing, we first get

$$\begin{aligned}\mathcal{A}_4 &= H(\bar{y},\bar{t},v(\bar{y},\bar{t}),-D_y\Phi) - H(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_x\Phi) \\ &\leq H(\bar{y},\bar{t},u(\bar{x},\bar{t}),-D_y\Phi) - H(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_x\Phi).\end{aligned}$$

Then, using **(H2)**, it follows that

$$\begin{aligned}\mathcal{A}_4 &\leq |H(\bar{y},\bar{t},u(\bar{x},\bar{t}),-D_y\Phi) - H(\bar{y},\bar{t},u(\bar{x},\bar{t}),D_x\Phi)| \\ &\qquad +|H(\bar{y},\bar{t},u(\bar{x},\bar{t}),D_x\Phi) - H(\bar{x},\bar{t},u(\bar{x},\bar{t}),D_x\Phi)| \\ &\leq m\left((1+|\bar{y}|)|D_x\Phi + D_y\Phi|\right) + m\left((1+|D_x\Phi|)|\bar{x}-\bar{y}|\right).\end{aligned}$$

Because of the properties of a modulus of continuity, for every $r > 0$ and every $\gamma > 0$, there exists a positive constant $C(\gamma)$ such that $m(r) \leq \gamma + C(\gamma)r$. Taking $\gamma = \eta/2$ and up to replace at each step $C(\eta)$ by a larger one which depends only on η, we obtain

$$\begin{aligned}\mathcal{A}_4 &\leq \frac{\eta}{2} + C(\eta)\Big((1+|\bar{y}|)|D_x\Phi + D_y\Phi| + (1+|D_x\Phi|)|\bar{x}-\bar{y}|\Big) \\ &\leq \frac{\eta}{2} + C(\eta)\left(\mathrm{e}^{L\bar{t}}(1+|\bar{y}|)|DK|(\varphi+\alpha) + (1+\mathrm{e}^{L\bar{t}}(|DK|(\varphi+\alpha)+K|D\varphi|))|\bar{x}-\bar{y}|\right) \\ &\leq \frac{\eta}{2} + C(\eta)\left(\mathrm{e}^{L\bar{t}}K(\varphi+\alpha) + |\bar{x}-\bar{y}|\right).\end{aligned}$$

But from Young's inequality, we have

$$|\bar{x}-\bar{y}| \leq \frac{|\bar{x}-\bar{y}|}{\varepsilon^p} + \varepsilon^q \leq K(\varphi+\alpha) + \varepsilon^q$$

and finally, we obtain

$$\mathcal{A}_4 \leq \eta + C(\eta)\left(\mathrm{e}^{L\bar{t}} K(\varphi+\alpha) + \varepsilon^q\right), \tag{33}$$

where $C(\eta)$ is a positive constant independent of α, ε and L.

10. End of the case $\bar{t} > 0$.
Plugging estimates (21), (24), (32) and (33) in (20) yields

$$\eta + L\Phi \ \leq \ \frac{\eta}{2} + (C + C(\eta))\Phi + C\,\mathrm{e}^{3L\bar{t}}\ \varepsilon^{-p/(p-1)}\ \alpha^\theta\ \alpha K + C(\eta)\varepsilon^q,$$

where C is a positive constant which is independent of ε, α, η and L. Thus, we can first choose a constant L large enough, namely $L(\eta) \geq C + C(\eta) + 1$, such that the previous inequality becomes

$$\frac{\eta}{2} + \Phi \leq C\,\mathrm{e}^{3L\bar{t}}\ \varepsilon^{-p/(p-1)}\ \alpha^\theta\ \alpha K + C(\eta)\,\varepsilon^q, \tag{34}$$

for every positive ε, α and η. Then we take $\alpha = \varepsilon^\ell$, choosing ℓ large enough such that

$$\theta\ell > \frac{p}{p-1}. \tag{35}$$

Thus (34) reads

$$\frac{\eta}{2} + \mathrm{e}^{L\bar{t}} K(\varphi+\alpha) \leq C\,\mathrm{e}^{3LT}\ \varepsilon^{\theta\ell - p/(p-1)}\ \alpha K + C(\eta)\varepsilon^q.$$

Taking ε small enough such that $C(\eta)\varepsilon^q < \eta/2$ and $C\,\mathrm{e}^{2LT}\ \varepsilon^{\theta\ell-p/(p-1)} \leq 1$, we obtain a contradiction. Finally, the conclusion of the preceding steps is: if we choose L, p and k as above and take $\alpha = \varepsilon^\ell$ (where ℓ satisfies (35)), then necessarily $\bar{t} = 0$ for sufficiently small ε.

11. From the previous step, we know that the maximum of the function $u - v - \Phi$ is achieved at a point $(\bar{x}, \bar{y}, 0)$ which implies, from (13) and Lemma 4.1, that there exists $\delta > 0$ such that

$$0 < \delta \leq u(\tilde{x},\tilde{t}) - v(\tilde{x},\tilde{t}) - \varepsilon^\ell\,\mathrm{e}^{L\tilde{t}} K(2\tilde{x}) - \eta\tilde{t} \leq u(\bar{x},0) - v(\bar{y},0) - K(\bar{x}+\bar{y})(\varphi+\varepsilon^\ell),$$

for ε and η small enough. Since $u(\bar{x},0) \leq u_0 \leq v(\bar{y},0)$, this inequality leads to

$$\delta \leq u_0(\bar{x}) - u_0(\bar{y}) - K(\bar{x}+\bar{y})(\varphi+\varepsilon^\ell).$$

Now, from **(H3-u_0)**(ν), there exists $C(\delta)>0$ such that

$$\begin{aligned}\delta &\le m\left((1+|\bar x|+|\bar y|)^\nu|\bar x-\bar y|\right)-K(\bar x+\bar y)(\varphi+\varepsilon^\ell)\\ &\le \frac{\delta}{2}+C(\delta)(1+|\bar x|+|\bar y|)^\nu|\bar x-\bar y|-K(\bar x+\bar y)(\varphi(\bar x-\bar y)+\varepsilon^\ell). \qquad (36)\end{aligned}$$

Since $|\bar x-\bar y|\le 1$, there exists $\tilde C$ such that $(1+|\bar x|+|\bar y|)^\nu\le \tilde C(1+|\bar x+\bar y|^\nu)$ and therefore (36) yields

$$\frac{\delta}{2}\le C(\delta)\tilde C(1+|\bar x+\bar y|^\nu)|\bar x-\bar y|-(1+|\bar x+\bar y|^k)\left(\frac{|\bar x-\bar y|^p}{\varepsilon^p}+\varepsilon^\ell\right). \qquad (37)$$

In order to study the right-hand side of this inequality, we introduce the function g defined by

$$g(r,s)=C(\delta)\tilde C(1+r^\nu)s-(1+r^k)\left(\frac{s^p}{\varepsilon^p}+\varepsilon^\ell\right),$$

for every $r,s\ge 0$. One easily shows that there exists a constant C depending only on $C(\delta),\tilde C$ and p such that

$$g(r,s)\le C\frac{(1+r^\nu)^{\frac{p}{p-1}}}{(1+r^k)^{\frac{1}{p-1}}}\varepsilon^{\frac{p}{p-1}}-\varepsilon^\ell(1+r^k). \qquad (38)$$

Then, using that, for every $r>0$, $\dfrac{(1+r^\nu)^{\frac{p}{p-1}}}{(1+r^k)^{\frac{1}{p-1}}}\le \tilde C(1+r^{\frac{\nu p-k}{p-1}})$, with $\tilde C=\tilde C(N,\nu,p,k)$, we obtain from (38)

$$g(r,s)\quad\le\quad f(r):=C\tilde C\left(1+r^{\frac{\nu p-k}{p-1}}\right)\varepsilon^{\frac{p}{p-1}}-(1+r^k)\varepsilon^\ell, \qquad (39)$$

for every $r,s\ge 0$. We take p large enough such that

$$\nu p-k>0. \qquad (40)$$

Since, in addition, we have already chosen $k>\nu+1$ at the beginning, the function f achieves a maximum at

$$r=C'\varepsilon^{\frac{\ell(p-1)-p}{p(\nu-k)}},$$

where the C' depends only on $C,\tilde C,p,\nu,k$. Replacing r by this value, we obtain that for every $r\ge 0$,

$$f(r)\le C\tilde C\varepsilon^{\frac{p}{p-1}}+\bar C\varepsilon^\gamma, \qquad (41)$$

where $\bar C$ is a constant depending on $C,\tilde C,p,\nu,k$ and

$$\gamma=\frac{kp-\ell(\nu p-k)}{p(k-\nu)}.$$

From (37),(39) and (41), we obtain

$$\frac{\delta}{2} \leq C\left(\varepsilon^{\frac{p}{p-1}} + \varepsilon^{\gamma}\right)$$

with C is a constant depending only on δ, ν, p, k and N.

In order to conclude through the above inequality by letting ε tend to 0, we need to have $\gamma > 0$ i.e. to be able to choose the parameters in such a way to have $\ell < \dfrac{kp}{\nu p - k}$ since $\nu p - k > 0$ from (40). To do so, we recall that we have chosen $k > \max\{2, \nu+1\}$ at the beginning, $p > k+1$ (see (28)) and $\ell > \dfrac{p}{\theta(p-1)}$ (see (35)) where $\theta = \dfrac{p-k-1}{(p-1)(k-1)}$ (see (29)). Thus, we need to find ℓ, k and p such that

$$k > \max\{2, \nu+1\}, \quad p > k+1, \quad \nu p > k \quad \text{and} \quad \frac{p(k-1)}{p-k-1} < \ell < \frac{kp}{\nu p - k}. \tag{42}$$

To fulfill the last condition in (42), it is sufficient to have $p(\nu - k(\nu-1)) > 2k$. It follows that, if we can find a suitable k, up to take p large enough, then all the conditions would be satisfied. We distinguish two cases:
— 1^{st} case. If $\nu \leq 1$, then we take $k > 2$ and we are done.
— 2^{nd} case. If $\nu > 1$, then we have to find k such that

$$\nu + 1 < k < \frac{\nu}{\nu - 1}.$$

It leads to a condition on ν, namely $\nu^2 < \nu + 1$ which is automatically satisfied provided $\nu < \dfrac{1+\sqrt{5}}{2}$ as we supposed it.

Finally, in any case, it is possible to fulfill conditions (42); thus the proof of the theorem is complete. □

Now we turn to the proof of the Lemma.

Proof of Lemma 4.1. We first prove (14). We are going to do it only for the subsolution u, the proof for the supersolution v being analogous.

To do so, we introduce for C, L, $\varepsilon > 0$, $k \geq 2$ and the sequence of smooth functions $(\chi_\varepsilon)_\varepsilon$ defined by

$$\chi_\varepsilon(x,t) = \mathrm{e}^{Lt}\left(C(1+|x|^2)^{\frac{\nu+1}{2}} + \varepsilon(1+|x|^k)\right).$$

Tedious but straightforward computations show that, for any C, ν and k, if L is chosen large enough, then χ_ε is a strict supersolution of (2) for any ε small enough.

On an other hand, since $u(x,0) \leq u_0(x)$ in $\mathbb{R}^N$ and since u_0 satisfies **(H3-u_0)**(ν), it is clear that if C is chosen large enough, then $u(x,0) \leq \chi_\varepsilon(x,0)$ in $\mathbb{R}^N$.

Finally since u is in $\mathcal{C}_{\mathrm{poly}}$, we have, for k large enough $u(x,t)/(1+|x|)^k \to 0$ when $|x| \to +\infty$ uniformly for $t \in [0,T]$ and therefore $u(x,t) \le \chi_\varepsilon(x,t)$ for $|x|$ large enough.

Using these three properties, one shows easily that $u(x,t) \le \chi_\varepsilon(x,t)$ in $I\!R^N \times [0,T]$: indeed, because of the last one, $u - \chi_\varepsilon$ achieves its maximum on $I\!R^N \times [0,T]$ but since u is a viscosity subsolution of (2) and χ_ε is a strict smooth supersolution of this equation, this maximum cannot be achieved for $t > 0$. Therefore it is achieved for $t = 0$ and the maximum is therefore negative, proving the claim. Letting ε tend to 0 yields (14).

Now we prove the second part of the lemma. Since $u(x,t) - v(y,t) - \Phi(x,y,t)$ is upper-semicontinuous, in order to prove that the supremum is achieved, it suffices to prove that this function tends to $-\infty$ when $|x|, |y| \to +\infty$ (uniformly with respect to $t \in [0,T]$). But

$$\begin{aligned} \mathrm{e}^{Lt} K(x+y)\left(\varphi(x-y)+\alpha\right) + \eta t &\ge \frac{|x-y|^p}{\varepsilon^p} + \alpha(1+|x+y|^k) \\ &\ge \min\left\{\alpha, \frac{1}{\varepsilon^p}\right\}\left(|x-y|^p + 1 + |x+y|^k\right). \end{aligned} \tag{43}$$

Since $p \ge k > \max\{2, \nu+1\}$, using the convexity of $r \mapsto r^k$, we have

$$|x-y|^p + 1 + |x+y|^k \ge |x-y|^k + |x+y|^k \ge |x|^k + |y|^k.$$

From (43) and (14), it follows

$$\begin{aligned} u(x,t) - v(y,t) - \Phi(x,y,t) &\le u(x,t) - v(y,t) - \min\left\{\alpha, \frac{1}{\varepsilon^p}\right\}\left(|x|^k + |y|^k\right) \\ &\le C\left(1 + |x|^{\nu+1} + |y|^{\nu+1}\right) - \min\left\{\alpha, \frac{1}{\varepsilon^p}\right\}\left(|x|^k + |y|^k\right) \end{aligned}$$

which proves the claim since $\nu + 1 < k$.

On an other hand, we have

$$u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - \Phi(\bar{x},\bar{y},\bar{t}) \ge u(\tilde{x},\tilde{t}) - v(\tilde{x},\tilde{t}) - \mathrm{e}^{L\tilde{t}} K(2\tilde{x})\alpha - \eta\tilde{t}.$$

From (13), the right-hand side is positive if α and η are sufficiently small.

For α and η sufficiently small, we have

$$0 < u(\bar{x},\bar{t}) - v(\bar{y},\bar{t}) - \Phi(\bar{x},\bar{y},\bar{t}) \le C\left(1 + |\bar{x}|^{\nu+1} + |\bar{y}|^{\nu+1}\right) - \mathrm{e}^{L\bar{t}} K(\bar{x}+\bar{y})(\varphi+\alpha) - \eta\bar{t}.$$

By using the convexity of $r \mapsto r^{\nu+1}$, it follows

$$\left(1 + |\bar{x}+\bar{y}|^k\right)\left(\frac{|\bar{x}-\bar{y}|^p}{\varepsilon^p} + \alpha\right) \le C\left(1 + |\bar{x}+\bar{y}|^{\nu+1} + |\bar{x}-\bar{y}|^{\nu+1}\right).$$

Since $\nu + 1 < k$, we obtain

$$\frac{|\bar{x}-\bar{y}|^p}{\varepsilon^p} \le C\left(1 + |\bar{x}-\bar{y}|^{\nu+1}\right)$$

which implies that $|\bar{x}-\bar{y}| \le 1$ for ε small enough since $p > \nu + 1$. And the proof of the lemma is complete. □

5 Proof of Corollary 2.1

The uniqueness part in the statement of Corollary 2.1 is an immediate consequence of Theorem 2.1. In this section, we prove the existence result using Perron's method. For the description of this method in the context of viscosity solutions, we refer to Ishii [17] and Crandall, Ishii and Lions [9].

A straightforward computation shows that, if $C > 0$ and L are large enough, then $\underline{u}(x,t) = -C\,\mathrm{e}^{Lt}(1+|x|^2)^{\frac{\nu+1}{2}}$ and $\overline{u}(x,t) = C\,\mathrm{e}^{Lt}(1+|x|^2)^{\frac{\nu+1}{2}}$ are respectively viscosity sub- and supersolution of (2).

We define the set $\mathcal{S}$ in the following way : an locally bounded, possibly discontinuous, function w defined on $I\!R^N \times [0,T]$ is in $\mathcal{S}$ if $\underline{u} \leq w \leq \overline{u}$ on $I\!R^N \times [0,T]$ and if w^* is a viscosity subsolution of (2) satisfying the initial condition in the viscosity sense, i.e.

$$\min\left\{\frac{\partial w}{\partial t} - \mathrm{Tr}\left[b(x,0,Dw)D^2w\right] + H(x,0,w,Dw), \quad w^*(x,0) - u_0(x)\right\} \leq 0 \tag{44}$$

in $I\!R^N \times \{0\}$. Then we set

$$u(x,t) = \sup_{w\in\mathcal{S}} w(x,t).$$

Classical arguments show that u is still a subsolution. Moreover, when a subsolution w satisfies the initial condition (44), it is well-known (see [1] for instance) that $w^*(x,0) \leq u_0(x)$ for all $x \in I\!R^N$. From the definition of u, it follows $u^*(x,0) \leq u_0(x)$ for all $x \in I\!R^N$.

From Perron's method, u is a supersolution which satisfies the initial condition (44) where we replace "min" by "max", "w^*" by "w_*" and "$\leq$" by "$\geq$". As above, it follows $u_*(x,0) \geq u_0(x)$ for all $x \in I\!R^N$.

Finally, we have $u^*(\cdot,0) \leq u_0 \leq u_*(\cdot,0)$ in $I\!R^N$. Applying our comparison result (Theorem 2.1) to u^* and u_*, we obtain $u^* \leq u_*$ which means that u is a continuous viscosity solution of (2) with initial datum u_0. □

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