Non fattening condition for the generalized evolution by mean curvature and applications

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Abstract

We prove a non fattening condition for a geometric evolution described by the level set approach. This condition is close to those of Soner [21] and Barles, Soner and Souganidis [5] but we apply it to some unbounded hypersurfaces. It allows us to prove uniqueness for the mean curvature equation for graphs with convex at infinity initial data, without any restriction on its growth at infinity, by seeing the evolution of the graph of a solution as a geometric motion.

1 Introduction

We consider the evolution Γ_t of a given initial hypersurface Γ_0 of $\mathbb{R}^{N+1}$ moving according to the normal velocity

$$\mathcal{V}_{(x,t)} = h(n_x, Dn_x), \tag{1}$$

where n_x and Dn_x stand respectively for an oriented unit normal and the second fundamental form of Γ_t at $x \in \Gamma_t$ and h is the given evolution law. The hypothesis on h will be introduced later but the key assumption in this paper is the map h to be elliptic with respect to the second variable. Namely, if X, Y are symmetric matrices, then

$$X \leq Y \quad \Longrightarrow \quad h(n_x, X) \geq h(n_x, Y). \tag{2}$$

The most typical example we are interested in is the celebrated mean curvature evolution where

$$\mathcal{V}_{(x,t)} = h(Dn_x) = -\mathrm{Tr}(Dn_x). \tag{3}$$

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To describe the evolution of Γ_t according to (1), different ways have been proposed; see the book of Giga [13]. Here, we follow the level set approach introduced by Barles [1] and Osher and Sethian [20], developed first independently by Evans and Spruck [12] and Chen, Giga and Goto [8].

The level set approach presents the advantage to be defined for all time $t \geq 0$, even past some singularities. We refer the reader to Section 2 for the definition and recall here that the evolution Γ_t by the level set approach is given, at each time t, as the 0-level set of an auxiliary function, namely $\Gamma_t := \{z \in \mathbb{R}^{N+1} : v(z,t) = 0\}$, where $v : \mathbb{R}^{N+1} \times [0, +\infty) \to \mathbb{R}$ is the solution of a suitable parabolic partial differential equation under the form

$$\frac{\partial v}{\partial t} + F(Dv, D^2 v) = 0 \quad \text{in} \quad \mathbb{R}^{N+1} \times (0, +\infty). \tag{4}$$

In this approach, one of the main issues is the so-called *fattening phenomenon* which happens when the front $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ has nonempty interior in $\mathbb{R}^{N+1} \times [0, +\infty)$. Some examples are known for which such a phenomenon arises (see Ilmanen [16] or Soner [21]). This fattening phenomenon is closely related to the non-uniqueness of the geometrical evolution (1). We refer to Ilmanen [16], Soner [21], Barles, Soner and Souganidis [5] or Barles, Biton and Ley [3] for further details.

Nevertheless, in [21] and [5] (see also [22] and [19]), the authors give sufficient conditions such that the front never fattens. For instance, Soner proves that compact hypersurfaces which are strictly starshaped never fatten for evolutions with normal velocity of curvature type given (see Section 4.1).

Our aim in this article is to extend this method to unbounded sets which are entire graphs of functions from $\mathbb{R}^N$ into $\mathbb{R}$. Even if Soner's condition could be applied for some unbounded hypersurfaces (like convex graphs), it does not hold for the case we have in mind (graphs which are convex at infinity, see below). In [5], the authors give a more general condition for C^2 hypersurfaces but it is not clear how to extend it to unbounded cases.

To be more specific on our result, let Γ_0 be the boundary of an open subset Ω_0 of $\mathbb{R}^{N+1}$ (notice that Γ_0 has empty interior). We prove that under suitable assumptions on the nonlinearity F appearing in (4), the front never fattens if there exists a family of affine dilations $(\mathcal{A}_\varepsilon)_{\varepsilon>0}$ going to identity as ε goes to 0 and such that

$$\mathrm{d}(\Gamma_0, \mathcal{A}_\varepsilon(\Gamma_0)) := \inf\{|a-b| : (a,b) \in \Gamma_0 \times \mathcal{A}_\varepsilon(\Gamma_0)\} \text{ is positive for any } \varepsilon > 0. \tag{5}$$

This condition is close to the one of [5] but is stated in a more readable way which does not require the initial set Γ_0 to be C^2.

Our main contribution is to show that this condition can be used to prove the uniqueness of the evolution by mean curvature of entire graphs which are convex at infinity. We say that a continuous function $f : \mathbb{R}^N \to \mathbb{R}$ is *convex at infinity* if there exists $R > 0$ such that, for any convex set $\mathcal{C} \subset \mathbb{R}^N \backslash B(0,R)$, the restriction $f : \mathcal{C} \to \mathbb{R}$ is convex. Our result is the following:

Theorem 1.1 *For any continuous initial data $u_0 : \mathbb{R}^N \to \mathbb{R}$ which is convex at infinity (without any growth restriction), there exists a unique solution of the mean curvature equation for graphs*

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2uDu, Du\rangle}{1+|Du|^2} = 0 \quad in \quad \mathbb{R}^N \times (0,+\infty), \tag{6}$$

with $u(\cdot,0) = u_0$.

The existence of a smooth solution $u \in C(\mathbb{R}^N \times [0,+\infty)) \cap C^\infty(\mathbb{R}^N \times (0,+\infty))$ was proved by Ecker and Huisken [11]. The very surprising fact is that this result holds without any growth restriction at infinity.

The question of uniqueness of these solutions without growth restriction at infinity is still open in the whole generality. Several partial results are known: In dimension $N = 1$, the problem was completely solved independently by Chou and Kwong [9] and in [4]; In any dimension, uniqueness was proved in the following situations: with polynomial-type restrictions on the growth of u_0 in [2], when u_0 is radially symmetric in $\mathbb{R}^N$ in [7] and when u_0 is convex in $\mathbb{R}^N$ in [3]. After this paper was completed we learned that Ishii and Mikami had obtained in [18] an uniqueness result for the motion of a graph by R-curvature under some convexity at infinity-type condition.

One could think that Theorem 1.1 is an easy generalization of the latter case but we point out that a very small perturbation of u_0 even on a compact set can modify the behaviour of the solution everywhere.

The uniqueness result of Theorem 1.1 holds in fact for more general quasilinear equations the class of which is described in [3] (see Section 4.2). Moreover, we give an example (see Remark 4.1) of application of (5) to initial data which are not convex at infinity. It follows first that the set of convex at infinity functions is not the right class of uniqueness for equations like (6). Secondly, it emphasizes that our condition is of geometrical nature in the sense that we do not have any idea of how to prove such a result by pdes' methods.

The paper is organized as follows. In Section 2, we briefly recall the level-set approach. In Section 3, we state and prove the sufficient condition (5). The last section is devoted to the proof of Theorem 1.1 and to its extension to other motions.

2 Preliminary about the level set approach

In this section, we recall what we need about the level set approach and give the definition of the generalized evolution Γ_t. For more details see the book [13].

We start by introducing some definitions and notations. Given an open subset Ω_0^+ of $\mathbb{R}^{N+1}$, we say that $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ is an *admissible partition* if $\Gamma_0 = \partial\Omega_0^+$ (∂ denotes the topological boundary) and $\Omega_0^- = \mathbb{R}^{N+1} \backslash (\Gamma_0 \cup \Omega_0^+)$. Notice that Γ_0 has an empty interior.

If $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ is an admissible partition, then the *signed distance* $d_s(\cdot, \Gamma_0)$ to Γ_0 is defined by

$$d_s(z, \Gamma_0) := \begin{cases} \mathrm{d}(z, \Gamma_0) & \text{if } z \in \Omega_0^+, \\ 0 & \text{if } z \in \Gamma_0, \\ -\mathrm{d}(z, \Gamma_0) & \text{if } z \in \Omega_0^-, \end{cases}$$

where d is the usual nonnegative distance in $\mathbb{R}^{N+1}$. Clearly $d_s(\cdot, \Gamma_0) \in UC(\mathbb{R}^{N+1})$, where "$UC$" denotes the uniformly continuous functions.

We aim at defining an evolution $(\Gamma_t, \Omega_t^+, \Omega_t^-)_{t\geq 0}$ starting from $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ where Γ_t evolves with normal velocity (1). Looking for an auxiliary function $v : \mathbb{R}^{N+1} \times [0, +\infty) \to \mathbb{R}$ satisfying, for every $t \geq 0$, the conditions

$$\{v(\cdot, t) = 0\} = \Gamma_t \quad \{v(\cdot, t) > 0\} = \Omega_t^+ \quad \text{and} \quad \{v(\cdot, t) < 0\} = \Omega_t^-,$$

we obtain that v has to be a solution, at least formally, of the so-called *level set equation* for (1),

$$\begin{cases} \dfrac{\partial v}{\partial t} + F(Dv, D^2 v) = 0 & \text{in} \quad \mathbb{R}^{N+1} \times (0, +\infty), \\ v(\cdot, 0) = v_0 & \text{in} \quad \mathbb{R}^{N+1}, \end{cases} \tag{7}$$

where, for instance $v_0 = d_s(\cdot, \Gamma_0)$, and

$$F(p, X) = -|p|\, h\left(-\frac{p}{|p|}, \left(-\frac{1}{|p|}(Id - \frac{p \otimes p}{|p|^2})X\right)_{|p^\perp}\right), \tag{8}$$

for $p \in D(F) \subset \mathbb{R}^{N+1}$ and $X \in \mathcal{S}_{N+1}$. Here and in the sequel, $\mathcal{S}_{N+1}$ denotes the space of symmetric matrices of size $N+1$ and $M_{|p^\perp}$ is the restriction to the subspace $p^\perp$ of the linear map induced by $M \in \mathcal{S}_{N+1}$. Note that in general, F has singularities in the gradient variable and $D(F) \neq \mathbb{R}^{N+1}$.

From the very definition of F, it follows that

$$F(\lambda p, \mu p \otimes p + \lambda M) = \lambda F(p, M) \quad \text{for all } p \in D(F), M \in \mathcal{S}_{N+1}, \lambda \geq 0, \mu \in \mathbb{R}, \tag{9}$$

and from (2), we have

$$M \geq N \implies F(p, M) \leq F(p, N) \quad \text{for all } p \in D(F), M, N \in \mathcal{S}_{N+1}. \tag{10}$$

It is worth noticing that (9) implies that the equation in (7) is invariant under changes of funtion $v \to \varphi \circ v$ with $\varphi' > 0$ whence we can work with bounded solutions of (7).

Moreover (2)-(10) imply that (7) is degenerate elliptic and a maximum principle is expected. To avoid technicalities, we state the comparison principle as an assumption:

(H) If v (respectively w) be a bounded uniformly continuous viscosity subsolution (respectively supersolution) of (7)-(8) satisfying $v(\cdot, 0) \leq w(\cdot, 0)$, then $v \leq w$ in $\mathbb{R}^{N+1} \times [0, +\infty)$.

Assumptions on F (or equivalently on h) which lead to this comparison principles are discussed in Section 4. We refer to [10] for a general discussion of the theory of viscosity solutions.

Now, we can state the following theorem and define the generalized evolution of Γ_0 with normal velocity given by (1):

Theorem 2.1 *Suppose that (2) and* **(H)** *hold. Then, for any $v_0 \in UC(\mathbb{R}^{N+1})$, there exists a unique UC viscosity solution of (7). Moreover, set*

$$(\Gamma_0, \Omega_0^+, \Omega_0^-) := \big(\{v_0 = 0\}, \{v_0 > 0\}, \{v_0 < 0\}\big) \tag{11}$$

and consider

$$(\Gamma_t, \Omega_t^+, \Omega_t^-) := \big(\{v(\cdot,t) = 0\}, \{v(\cdot,t) > 0\}, \{v(\cdot,t) < 0\}\big).$$

Then the family $(\Gamma_t, \Omega_t^+, \Omega_t^-)_{t\geq 0}$ is independent of the choice of $v_0 \in UC(\mathbb{R}^{N+1})$ satisfying (11). Hence, it allows to define $(\Gamma_t)_{t\geq 0}$ as the generalized evolution of Γ_0 with normal velocity (1) starting from the initial admissible partition $(\Gamma_0, \Omega_0^+, \Omega_0^-)$.

We do not prove the theorem here. Various assumptions on F (or h) can be required in order that **(H)** and the theorem holds true. In their celebrated papers, Evans and Spruck [12] and Chen, Giga and Goto [8] proved in particular Theorem 2.1 for the mean curvature evolution (3). In that case, the level set equation (7) is the well-known mean curvature equation,

$$\frac{\partial v}{\partial t} - \Delta v + \frac{\langle D^2 v Dv, Dv\rangle}{|Dv|^2} = 0 \quad \text{in } \mathbb{R}^{N+1} \times (0, +\infty).$$

Other cases are treated in Giga, Goto, Ishii and Sato [14], Barles, Souganidis and Soner [5], Soner [21], Ishii and Souganidis [19], Ishii [17], Souganidis [23], Giga and Sato [15], Barles, Biton and Ley [3]. See the book of Giga [13] and Section 4 for explicit examples.

3 A sufficient condition for the non fattening

In this section we give a condition on an initial hypersurface Γ_0 under which its generalized evolution never fattens. To this end we need the following assumption: there exists a positive continuous real-valued function m such that $m(1) = 1$ and

$$F(p, \lambda M) = m(\lambda) F(p, M) \quad \text{for all } p \in D(F), M \in \mathcal{S}_{N+1}, \lambda > 0, \tag{12}$$

where F is defined by (8). Let $\mathcal{T}$ be the group of affine dilations of $\mathbb{R}^{N+1}$,

$$\mathcal{T} = \left\{ \mathcal{A} : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1} : \mathcal{A}(z) = \lambda z + z_0, \lambda \in \mathbb{R}\backslash\{0\}, z_0 \in \mathbb{R}^{N+1} \right\}.$$

Lemma 3.1 *Assume that* **(H)** *and (12) hold. Suppose that $D(F)$ is invariant under dilations, that $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ is an admissible initial partition and $\mathcal{A} \in \mathcal{T}$ with coefficient $\lambda \neq 0$. Let v (respectively $v_{\mathcal{A}}$) be the solution of (7) associated to $d_s(\cdot, \Gamma_0)$ (respectively $d_s(\cdot, \mathcal{A}(\Gamma_0))$). Then we have*

$$v_{\mathcal{A}}(z,t) = \lambda\, v\left(\mathcal{A}^{-1}z, \frac{t}{\lambda m(\lambda)}\right).$$

Proof of Lemma 3.1. Suppose that $\mathcal{A}z = \lambda z + z_0$. Let $v_0 = d_s(\cdot, \Gamma_0)$, (respectively $w_0 = d_s(\cdot, \mathcal{A}(\Gamma_0))$) and v (respectively w) be the unique uniformly continuous solution of (7) with initial data v_0 (respectively w_0). Observing that $w_0 = \lambda v_0 \circ \mathcal{A}^{-1}$ we set

$$w(z,t) = \lambda v\left(\frac{z - z_0}{\lambda}, \frac{t}{\lambda m(\lambda)}\right).$$

Using (9) and (12), one checks that w is a solution of (7) with initial data w_0. Therefore we get the Lemma by the uniqueness result for (7). □

Let A, B be two subsets of $\mathbb{R}^{N+1}$. We define the *minimum distance* $\mathrm{d}(A,B)$ between A and B by

$$\mathrm{d}(A,B) := \inf_{a\in A, b\in B} |a - b|.$$

Lemma 3.2 *For any admissible partitions $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ and $(\tilde{\Gamma}_0, \tilde{\Omega}_0^+, \tilde{\Omega}_0^-)$, if $\tilde{\Omega}_0^+ \cup \tilde{\Gamma}_0 \subset \Omega_0^+$ then*

$$\mathrm{d}(\Gamma_0, \tilde{\Gamma}_0) \geq \eta \geq 0 \quad \Longrightarrow \quad d_s(\cdot, \Gamma_0) \geq d_s(\cdot, \tilde{\Gamma}_0) + \eta.$$

Remark 3.1 A consequence of the comparison assumption **(H)** and Lemma 3.2 is an inclusion principle which roughly states that, if Ω_0^+ and $\tilde{\Omega}_0^+$ are such that $\Omega_0^+ \subset \tilde{\Omega}_0^+$, then this inclusion remains true for all time: $\Omega_t^+ \subset \tilde{\Omega}_t^+$. We point out that this inclusion principle is an important underlying property of geometrical evolutions satisfying (2). For a general study of geometrical evolutions which satisfies this principle, see Barles and Souganidis [6].

Proof of Lemma 3.2. We distinguish several cases according to the position of z (see Figure 1).

Case 1 when $z_1 \in \Omega_0^- \cup \Gamma_0$. Let $a = \mathrm{d}(z_1, \Gamma_0)$ and $b = \mathrm{d}(z_1, \tilde{\Gamma}_0) = |z_1 - \tilde{z}_1|$, with $\tilde{z}_1 \in \tilde{\Gamma}_0$. Consider a point $z_1' \in [z_1, \tilde{z}_1] \cap \Gamma_0$. We have

$$d_s(z_1, \Gamma_0) - d_s(z_1, \tilde{\Gamma}_0) = -a + b = -a + |z_1 - z_1'| + |z_1' - \tilde{z}_1|.$$

But $|z_1 - z_1'| \geq \mathrm{d}(z_1, \Gamma_0) = a$ and $|z_1' - \tilde{z}_1| \geq d_e(\Gamma_0, \tilde{\Gamma}_0) \geq \eta$; therefore $d_s(z_1, \Gamma_0) - d_s(z_1, \tilde{\Gamma}_0) \geq \eta$.

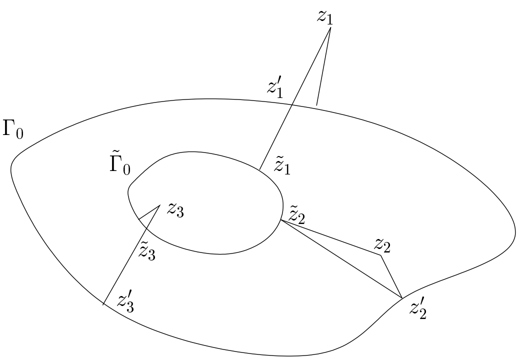

Figure 1: *The different cases under consideration in the proof of Lemma 3.2.*

Case 2 when $z_2 \in \Omega_0^+ \cap \tilde{\Omega}_0^-$. Let $a = \mathrm{d}(z_2, \Gamma_0) = |z_2 - z_2'|$ with $z_2' \in \Gamma_0$, and $b = \mathrm{d}(z_2, \tilde{\Gamma}_0) = |z_2 - \tilde{z}_2|$, with $\tilde{z}_2 \in \tilde{\Gamma}_0$. We have

$$d_s(z_2, \Gamma_0) - d_s(z_2, \tilde{\Gamma}_0) = a + b \geq |z_2' - \tilde{z}_2| \geq \eta.$$

Case 3 when $z_3 \in \tilde{\Omega}_0^- \cup \tilde{\Gamma}_0$. Let $a = \mathrm{d}(z_3, \Gamma_0) = |z_3 - z_3'|$ with $z_3' \in \Gamma_0$, and $b = \mathrm{d}(z_3, \tilde{\Gamma}_0)$. Consider a point $\tilde{z}_3 \in [z_3, z_3'] \cap \tilde{\Gamma}_0$. We have

$$d_s(z_3, \Gamma_0) - d_s(z_3, \tilde{\Gamma}_0) = a - b = |z_3 - \tilde{z}_3| + |\tilde{z}_3 - z_3'| - b.$$

But $|\tilde{z}_3 - z_3'| \geq d_e(\Gamma_0, \tilde{\Gamma}_0) \geq \eta$ and $|z_3 - \tilde{z}_3| \geq \mathrm{d}(z_3, \tilde{\Gamma}_0) = b$; thus $d_s(z_3, \Gamma_0) - d_s(z_3, \tilde{\Gamma}_0) \geq \eta$, which completes the proof of the Lemma 3.2. □

1

Now, we can state the main result of this section.

Theorem 3.1 *Let $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ be an admissible initial partition and assume that Theorem 2.1 and (12) hold. If there exists a family $(\mathcal{A}_\varepsilon)_{\varepsilon>0} \subset \mathcal{T}$ and a sequence of positive numbers $(\eta_\varepsilon)_{\varepsilon>0}$ such that*

$$\mathcal{A}_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} Id \quad \text{and} \quad \mathrm{d}(\Gamma_0, \mathcal{A}_\varepsilon(\Gamma_0)) \geq \eta_\varepsilon > 0 \ for\ \varepsilon > 0, \tag{13}$$

then the front $\bigcup_{t \geq 0} \Gamma_t \times \{t\}$ has empty interior in $\mathbb{R}^{N+1} \times [0, +\infty)$.

Proof of Theorem 3.1. Let v and $v_{\mathcal{A}_\varepsilon}$ be the uniformly continuous viscosity solutions of (7) associated to the initial data $d_s(\cdot, \Gamma_0)$ and $d_s(\cdot, \mathcal{A}_\varepsilon(\Gamma_0))$. From Lemma 3.1, for every $(z, t) \in \mathbb{R}^{N+1} \times [0, +\infty)$, we have

$$v_{\mathcal{A}_\varepsilon}(z, t) = \lambda_\varepsilon v \left(\mathcal{A}_\varepsilon^{-1} z, \frac{t}{\lambda_\varepsilon m(\lambda_\varepsilon)} \right),$$

where λ_ε is the coefficient of $\mathcal{A}_\varepsilon$. Next, from (13), Lemma 3.2 and the comparison principle **(H)**, we get that, for any $\varepsilon > 0$,

$$v(z,t) \geq v_{\mathcal{A}_\varepsilon}(z,t) + \eta_\varepsilon.$$

Therefore

$$v(z,t) \geq \lambda_\varepsilon v\left(\mathcal{A}_\varepsilon^{-1} z, \frac{t}{\lambda_\varepsilon m(\lambda_\varepsilon)}\right) + \eta_\varepsilon. \tag{14}$$

Assume now that the front $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ has nonempty interior in $\mathbb{R}^{N+1} \times [0,+\infty)$. It follows that there is some $(z_0, t_0) \in \mathbb{R}^{N+1} \times (0,+\infty)$ and some $r > 0$ such that

$$v \equiv 0 \quad \text{in} \quad B(z_0, r) \times [t_0 - r, t_0 + r].$$

Since $\mathcal{A}_\varepsilon \to Id$, one has $\lambda_\varepsilon \to 1$ and then, for ε sufficiently small,

$$\left(\mathcal{A}_\varepsilon^{-1} z_0, \frac{t_0}{\lambda_\varepsilon m(\lambda_\varepsilon)}\right) \in B((z_0,t_0), r) \times [t_0 - r, t_0 + r].$$

Writing (14) at the point (z_0, t_0), we obtain a contradiction which ends the proof. □

4 Application to uniqueness results.

In this section we give some applications to Theorem 3.1. The first application is known and concerns the evolution of compact sets. The second, which is the main result and the motivation of this work, gives new uniqueness results for quasilinear parabolic pdes.

We recall some explicit assumptions on the evolution law on h which appears in (1) or, equivalently, on F defined by (8) which imply the comparison assumption **(H)**.

4.1 Uniqueness of generalized evolutions of compact sets

We show that Theorem 3.1 applies to recover some results of [21] and [19]. We suppose first

(H1) The evolution law h is linear with respect to the second fundamental form, i.e, $h = -\mathrm{Tr}(G(n_x)Dn_x)$, and $G : S^N \to \mathcal{S}^+_{N+1}$ is continuous, where $S^N = \{\xi \in \mathbb{R}^{N+1} : |\xi| = 1\}$ is the unit sphere and $\mathcal{S}^+_{N+1}$ is the set of nonnegative symmetric matrices of size $N+1$.

Lemma 4.1 *[21] Under assumption* **(H1)**, **(H)** *holds.*

Noticing that (12) holds with $m(r) = r$, we have

Theorem 4.1 *[21] Let $(\Gamma_0, \Omega_0^+, \Omega_0^-)$ be an admissible partition such that Γ_0 has empty interior and evolves with velocity (1) satisfying* **(H1)**. *Suppose that $\overline{\Omega_0^+}$ is a compact subset which is strictly starshaped, namely: there exists $z_0 \in \Omega_0^+$ such that, for all $z \in \overline{\Omega_0^+}$, $[z_0, z[\subset \Omega_0^+$. Then $\displaystyle\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ has empty interior.*

This theorem was proved by Soner [21, Theorem 9.3]. Our proof is basically the same so we only sketch it. Up to a translation, we can suppose that $z_0 = 0$ and we check that the family $\mathcal{A}_\varepsilon(z) = (1-\varepsilon)z$, for $\varepsilon \in (0,1)$ satisfies (13). We conclude by Theorem 3.1. It is worth pointing out again that the previous result includes the mean curvature motion (3).

We present another example of motion, namely the motion by Gaussian curvature:

(H2) The evolution law h is given by $h(Dn_x) = \kappa_1^+\kappa_2^+\cdots\kappa_N^+$ where $\kappa_1, \kappa_2, \cdots, \kappa_N$ are the principal curvatures of Γ_t (the eigenvalues of Dn_x) and $r^+ := \max\{r, 0\}$.

In this case, the level set equation (7)-(8) reads

$$\frac{\partial v}{\partial t} - |Dv| \det{}_+ \left[\frac{1}{|Dv|}(Id - \frac{Dv\otimes Dv}{|Dv|^2})D^2v(Id - \frac{Dv\otimes Dv}{|Dv|^2}) + \frac{Dv\otimes Dv}{|Dv|^2}\right] = 0$$

in $\mathbb{R}^{N+1}\times(0,+\infty)$, where, for any symmetric matrix $X \in \mathcal{S}_{N+1}$ with eigenvalues $\lambda_1, \cdots \lambda_{N+1}$, $\det_+(X) = \lambda_1^+\cdots\lambda_{N+1}^+$. Under Assumption **(H2)**, **(H)** holds and therefore Theorem 2.1 applies (see [19]). Moreover (12) holds with $m(r) = r^N$. Thus Theorem 4.1 holds true with the same proof. By this way, we recover [19, Proposition 3.4] without assuming C^2 regularity for Γ_0.

4.2 Uniqueness of solutions of quasilinear parabolic pdes

We turn to our main application. Consider the following pde,

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{Tr}\left[b\,(Du)\,D^2u\right] = 0 \quad \text{in} \quad \mathbb{R}^N\times(0,+\infty), \\ u(\cdot,0) = u_0 \in C(\mathbb{R}^N), \end{cases} \tag{15}$$

where $b : \mathbb{R}^N \to \mathcal{S}_N^+$ and $\mathcal{S}_N^+$ is the set of the nonnegative symmetric matrices of size N. Note that Equation (15) is quasilinear parabolic (possibly degenerate). Existence of solutions to (15) is not the point here, we refer to [11], [9], [3] for quite general results for any continuous initial data u_0 without any growth restriction at infinity which is unusual. The question we address here is the uniqueness of these solutions

In [3] we show that under suitable assumptions on the diffusion matrix b (see below), the graphs of the solutions of (15) are hypersurfaces of $\mathbb{R}^{N+1}$ moving accordind to a geometrical law of type (1). It makes possible to define the generalized evolution Γ_t of

$$\Gamma_0 = \mathrm{Graph}(u_0) = \{(x, u_0(x)) : x \in \mathbb{R}^N\} \subset \mathbb{R}^{N+1},$$

and we prove that the graphs of all the continuous viscosity solutions of (15) are contained in the front $\bigcup_{t\geq 0}\Gamma_t\times\{t\}$. It follows that the uniqueness of continuous viscosity solutions is equivalent to the non fattening of the front (see [3, Theorem 6.2]).

In this case, the level-set equation is (7) with

$$F(Dv, D^2v) = -\mathrm{Tr}\left[b\left(-\frac{D_xv}{D_yv}\right)\left(D^2_{xx}v - 2D^2_{xy}\otimes\frac{D_xv}{D_yv} + D_{yy}v\frac{D_xv}{D_yv}\otimes\frac{D_xv}{D_yv}\right)\right], \quad (16)$$

and $D(F) = \{p = (p_1, p_2, \cdots, p_{N+1}) \in \mathbb{R}^{N+1} : p_{N+1} = 0\}$. The precise assumptions we need are

(H3) The map $b : \mathbb{R}^N \to \mathcal{S}_N^+$ is continuous, there exists a constant $C > 0$ such that $|b(q)| + |b(q)q| + |\langle b(q)q, q\rangle| \le C$ for all $q \in \mathbb{R}^N$ and there exists a continuous map $b_\infty : \{\xi \in \mathbb{R}^N : |\xi| = 1\} \to \mathcal{S}_N^+$ such that $b_\infty(q) = \lim_{\lambda\to\pm\infty} b(\lambda q)$.

Lemma 4.2 *[3] Under assumption* **(H3)***,* **(H)** *holds.*

Roughly speaking, these assumptions allow us to control the singularities of F in order to prove the comparison result and apply Theorem 2.1. Our main result is the following:

Theorem 4.2 *Assume that F defined by (16) satisfies* **(H3)***. If $u_0 \in C(\mathbb{R}^N)$ is convex at infinity, then (15) has at most one continuous viscosity solution.*

Before giving the proof of the theorem, we do some comments. The typical example we have in mind is

$$b(p) = Id - \frac{p\otimes p}{1+|p|^2}.$$

In this case, (15) reduces to the mean curvature equation for graphs (6) and therefore the above theorem includes as a particular case Theorem 1.1 which is the motivation of this work. For other examples, see [3]. A similar result for motion of a graph by R-curvature with convex at infinity initial data was obtained by Ishii and Mikami [18].

Remark 4.1 *A function u_0 which satisfies the assumptions of Theorem 3.1 but is not convex at infinity.* Let $f, g \in C(\mathbb{R}^N)$ such that f is convex in $\mathbb{R}^N$ and $g(x) \to 0$ as $|x| \to +\infty$. Set $u_0 = \max\{f(x), g(x)\}$. Then $u_0 \in C(\mathbb{R}^N)$ is not necessarily convex at infinity (in the case $N = 1$, take for instance f, g defined by $f(x) = \mathrm{e}^{x^3}$ and $g(x) = \sin x/x$). But u_0 satisfies (13). We do not give the proof since it is close to the proof of Theorem 4.2. This example shows that Theorem 3.1 applies to a larger class of graphs than convex at infinity ones.

Proof of Theorem 4.2. From [3, Theorem 6.2], it is sufficient to prove that the generalized evolution Γ_t of $\Gamma_0 := \mathrm{Graph}(u_0)$ does not develop an interior. We proceed in two steps.
Step 1. To emphasize the main ideas without technicality, we first suppose that u_0 is convex in $\mathbb{R}^N$.

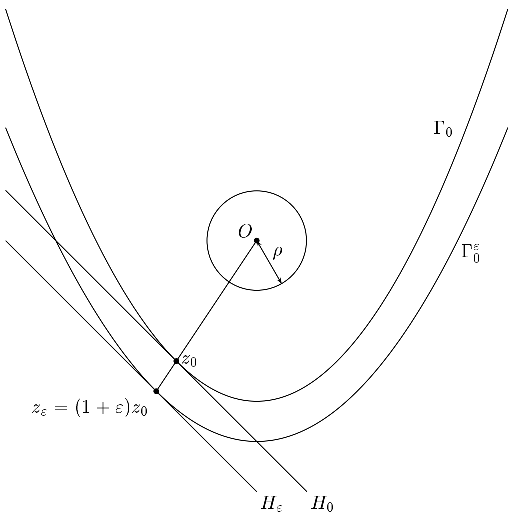

Figure 2: $\Gamma_0 = \text{Graph}(u_0)$ *with* u_0 *convex.*

Up to translate Γ_0, we can assume that there exists $\rho > 0$ such that $\overline{B(0,\rho)} \subset \text{Epi}(u_0) = \{(x,r) \in \mathbb{R}^{N+1} : r \geq u_0(x)\}$. Consider the family $(\mathcal{A}_\varepsilon)_{\varepsilon>0} \subset \mathcal{T}$, defined by

$$\mathcal{A}_\varepsilon = (1+\varepsilon)Id, \quad 0 < \varepsilon < 1,$$

and set $\Gamma_0^\varepsilon = \mathcal{A}_\varepsilon(\Gamma_0)$.

We aim at applying Theorem 3.1. We claim there exists a sequence $(\eta_\varepsilon)_{\varepsilon>0}$ of positive numbers such that

$$d(\Gamma_0, \mathcal{A}_\varepsilon(\Gamma_0)) \geq \eta_\varepsilon. \tag{17}$$

We prove this claim. Let $z_0 = (x_0, y_0) \in \Gamma_0$ and $z_\varepsilon = \mathcal{A}_\varepsilon(z_0) = (1+\varepsilon)z_0$. For ξ in the convex subdifferential $\partial u_0(x_0)$ of u_0 at x_0, consider the support hyperplane

$$H_0 = \{z = (x,y) \in \mathbb{R}^{N+1} : y = \langle \xi, x - x_0 \rangle + y_0\},$$

to $\text{Epi}(u_0)$ passing through z_0 (see Figure 2). Since u_0 is convex, $\text{Epi}(u_0)$ lies in the half-space $\{y \geq \langle \xi, x - x_0 \rangle + y_0\}$; it follows

$$\mathrm{d}(z_\varepsilon, \Gamma_0) \geq \mathrm{d}(z_\varepsilon, H_0) = \mathrm{d}(H_\varepsilon, H_0),$$

where $H_\varepsilon = \{z = (x,y) : y = \langle \xi, x - (1+\varepsilon)x_0 \rangle + (1+\varepsilon)y_0\}$ is the parallel hyperplane to H_0 passing through z_ε. Noticing that $H_\varepsilon = \mathcal{A}_\varepsilon(H_0)$, we obtain

$$\mathrm{d}(H_\varepsilon, H_0) \geq \varepsilon\, \mathrm{d}(0, H_0) \geq \rho\varepsilon.$$

Finally, for every $z_\varepsilon \in \Gamma_0^\varepsilon$, $\mathrm{d}(z_\varepsilon, \Gamma_0) \geq \rho\varepsilon$; therefore (17) holds with $\eta_\varepsilon = \rho\varepsilon$. It follows that assumption (13) of Theorem 3.1 holds and we obtain the desired conclusion.

Note that Step 1 provides a new proof of [3, Theorem 10.1].

Step 2. The general case.

From now on, we assume that the map u_0 is convex at infinity. Let $R_0 > 0$ be some constant such that u_0 is convex on any convex subset of $\mathbb{R}^N \backslash B(0, R_0)$. For $x \in \mathbb{R}^N$, with $|x| > R_0$, we define the subdifferential $\partial u_0(x)$ as the subdifferential at x of the restriction of u_0 to any convex neighbourhood of x contained in $\mathbb{R}^N \backslash B(0, R_0)$. Since the notion of subdifferential is local, $\partial u_0(x)$ is well defined. Let us point out that $\partial u_0(x)$ enjoys the following property: if $p \in \partial u_0(x)$, then

$$\forall y \in \mathbb{R}^N \text{ with } [x, y] \cap \overline{B(0, R_0)} = \emptyset, \quad u_0(y) \geq u_0(x) + \langle p, y - x \rangle \ .$$

Moreover $\partial u_0(x)$ is non empty as soon as $|x| > R_0$.

With this in mind, let us state a preliminary (and technical) lemma (for the proof and a comment, see below).

Lemma 4.3 *Assume that u_0 is convex at infinity. Then there is some radius $R > 0$ and some constant $c \in \mathbb{R}$ such that*

(i) u_0 is convex on any convex subset of $\mathbb{R}^N \backslash B(0, R)$,

(ii) for any $x \notin B(0, R + 1)$, for any $p \in \partial u_0(x)$,

$$u_0(x) - \langle p, x \rangle \leq -|p| + c \ . \tag{18}$$

For any $\varepsilon > 0$ and any $\lambda \in (0, 1)$, let us set

$$u_{\varepsilon,\lambda}(x) = (1 - \lambda) \left[u_0 \left(\frac{x}{1 - \lambda} \right) + \varepsilon \right] \ .$$

Let us underline that the graph of $u_{\varepsilon,\lambda}$ is the image of the graph of u_0 by the similitude $\mathcal{A}_{\varepsilon,\lambda}$ defined by

$$\mathcal{A}_{\varepsilon,\lambda}(z) = (1 - \lambda)(z + (0, \varepsilon)).$$

Note that $\mathcal{A}_{\varepsilon,\lambda} \to Id$ as $\varepsilon, \lambda \to 0$.

To conclude applying Theorem 3.1, it is sufficient to prove the following claim: *for any $\varepsilon > 0$, there is some $\lambda_\varepsilon \in (0, 1)$ such that, for any $\lambda \in (0, \lambda_\varepsilon)$,*

$$\mathrm{d}\left(\mathrm{Graph}(u_0), \mathrm{Graph}(u_{\varepsilon,\lambda})\right) \ > \ 0 \ .$$

Let R and c be as in Lemma 4.3. Without loss of generality, up to some translation, we can assume that $c = -1$. Thus we have

$$\forall x \notin B(0, R + 1), \ \forall p \in \partial u_0(x), \ u_0(x) - \langle p, x \rangle \leq -(1 + |p|) \ . \tag{19}$$

Since u_0 is continuous, we have

$$\forall z \in \mathbb{R}^N,\ \mathrm{d}((z, u_0(z)), \mathrm{Graph}(u_0) + (0, \varepsilon)) > 0\ .$$

Therefore, $\gamma_\varepsilon = \min_{|z| \le R+1} \mathrm{d}((z, u_0(z)), \mathrm{Graph}(u_0) + (0, \varepsilon))$ is positive.

Let $x, y \in \mathbb{R}^N$. We want to estimate from below $|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))|$ by some constant independant of x and y. For this, let us first assume that $x \in B(0, R+1)$. We can also suppose that $|x - y| \le 1$. Then

$$|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))| \ge |(x, u_0(x)) - (y, u_0(y) + \varepsilon)| - |(u_0(y) + \varepsilon) - (1-\lambda)(u_0(\frac{y}{1-\lambda}) + \varepsilon)|\ .$$

Since $(y, u_0(y) + \varepsilon)$ belongs to $\mathrm{Graph}(u_0) + (0, \varepsilon)$, we have

$$|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))| \ge \gamma_\varepsilon - \left[\lambda\varepsilon + \lambda|u_0(\frac{y}{1-\lambda})| + |u_0(y) - u_0(\frac{y}{1-\lambda})|\right]\ .$$

We can choose $\lambda_\varepsilon > 0$ small enough such that, for every $\lambda \in (0, \lambda_\varepsilon)$,

$$\forall y \in B(0, R+2),\quad \lambda\varepsilon + \lambda|u_0(\frac{y}{1-\lambda})| + |u_0(y) - u_0(\frac{y}{1-\lambda})| \le \gamma_\varepsilon/2\ .$$

This leads to

$$|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))| \ge \gamma_\varepsilon/2\ .$$

Let us now assume that $x \notin B(0, R+1)$. Let us choose some $p \in \partial u_0(x)$. Since $|y - x| \le 1$ and $|x| \ge R+1$, the segment $[x, y/(1-\lambda)]$ is a subset of $\mathbb{R}^N \setminus \overline{B(0, R)}$. Thus, by convexity

$$u_0(\frac{y}{1-\lambda}) \ge u_0(x) + \langle p, \frac{y}{1-\lambda} - x\rangle\ ,$$

which implies that

$$u_{\varepsilon,\lambda}(y) \ge (1-\lambda)[u_0(x) - \langle p, x\rangle + \varepsilon] + \langle p, y\rangle\ .$$

Let us define

$$\forall z \in \mathbb{R}^N,\ \pi(z) = (1-\lambda)[u_0(x) - \langle p, x\rangle + \varepsilon] + \langle p, z\rangle\ .$$

Let us notice that, on the one hand $\pi(x) \ge u_0(x)$ because of (19) and, on the another hand $u_{\varepsilon,\lambda}(y) \ge \pi(y)$. Therefore

$$|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))| \ge d((x, u_0(x)), \mathrm{Graph}(\pi)) = \gamma[1 + |p|^2]^{-\frac{1}{2}}\ ,$$

where $\gamma = \lambda(\langle p, x\rangle - u_0(x)) + (1-\lambda)\varepsilon$. Therefore, using (19), we get

$$|(x, u_0(x)) - (y, u_{\varepsilon,\lambda}(y))| \ge \lambda\frac{1 + |p|}{(1 + |p|^2)^{\frac{1}{2}}} \ge \frac{\lambda}{2}\ .$$

In conclusion, we have proved that, for any $\varepsilon > 0$ and any $\lambda \in (0, \lambda_\varepsilon)$,

$$\mathrm{d}\,(\mathrm{Graph}(u_0), \mathrm{Graph}(u_{\varepsilon,\lambda}))\ \ge\ \min\{\gamma_\varepsilon/2, \lambda/2\}\ > 0\ ,$$

which completes the proof of Theorem 4.2. □

Remark 4.2 Lemma 4.3 has the following geometric interpretation: Let $\mathcal{C}$ be any open convex subset of $\mathbb{R}^N \backslash B(0, R+1)$. Let $u_0^{\mathcal{C}}$ be the smallest convex function which coincides with u_0 on $\mathcal{C}$, namely

$$\forall x \in \mathbb{R}^N,\ u_0^{\mathcal{C}}(x) = \sup\{u_0(z) + \langle p, x - z\rangle \ :\ z \in \mathcal{C} \text{ and } p \in \partial u_0(z)\} \ .$$

Then inequality (18) states that $u_0^{\mathcal{C}}$ is bounded from above by the constant c on the ball $B(0, 1)$.

Proof of Lemma 4.3. Let $R_0 > 0$ be some constant such that u_0 is convex on any convex subset of $\mathbb{R}^N \backslash B(0, R_0)$. Let us fix $z \in \mathbb{R}^N$ with $|z| \leq 1$ and let us set $u_z(\cdot) = u_0(\cdot + z)$. Then u_z is convex on any convex subset of $\mathbb{R}^N \backslash B(0, R_0 + 1)$.

We claim that, for any $x \in \mathbb{R}^N$ with $|x| > R_0 + 2$, for any $p \in \partial u_z(x)$, for any $q \in \partial u_z(y)$ where $y = (R_0 + 2)x/|x|$, we have

$$u_z(x) - \langle p, x\rangle \ \leq \ u_z(y) - \langle q, y\rangle \ . \tag{20}$$

Indeed, since the segment $[x, y]$ has an empty intersection with $B(0, R_0 + 1)$, and since u_z is convex on any convex subset of $\mathbb{R}^N \backslash B(0, R_0 + 1)$, we have

$$u_z(y) \geq u_z(x) + \langle p, y - x\rangle \ . \tag{21}$$

Moreover, from the convexity of u_z on some convex neighbourhood of the segment $[x, y]$, we have $\langle p - q, x - y\rangle \geq 0$. Since $x - y = (|x|/(R_0 + 2) - 1)y$, with $|x| > R_0 + 2$, this implies that $\langle p, y\rangle \ \geq \ \langle q, y\rangle$. From this inequality and from (21) we deduce that

$$u_z(y) \geq u_z(x) - \langle p, x\rangle + \langle q, y\rangle \ ,$$

which proves our claim.

Since u_0 is convex on any convex subset of $\mathbb{R}^N \backslash B(0, R_0)$, u_0 is locally Lipschitz continuous on this set. Let L be some Lipschitz constant of u_0 on the set $B(0, R_0+3) \backslash B(0, R_0+1)$. In particular, for any $y \in B(0, R_0 + 3) \backslash B(0, R_0 + 1)$ and any $q \in \partial u_0(y)$, we have $|q| \leq L$.

Let us fix $z \in B(0, 1)$ and $x \in \mathbb{R}^N$ with $|x| \geq R_0 + 3$. We apply (20) to z and to $x - z$: (20) states that

$$\forall p \in \partial u_0(x),\ u_0(x) - \langle p, x - z\rangle \ \leq \ u_0(y + z) - \langle q, y\rangle$$

where $y = (R_0 + 2)(x - z)/|x - z|$ and $q \in \partial u_0(y + z)$. Let us notice that $|q| \leq L$ since $y + z \in B(0, R_0 + 3) \backslash B(0, R_0 + 1)$. Therefore

$$\forall p \in \partial u_0(x),\ u_0(x) - \langle p, x - z\rangle \ \leq \ \|u_0\|_{L^\infty(B(0,R_0+3))} + L(R_0 + 2) \ .$$

Since this inequality holds true for any z with $|z| \leq 1$, we finally deduce that

$$\forall p \in \partial u_0(x),\ u_0(x) - \langle p, x\rangle \leq c - |p| \ ,$$

with $c = \|u_0\|_{L^\infty(B(0,R_0+3))} + L(R_0 + 2)$. This is the desired result if we set $R = R_0 + 3$. $\square$

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