Uniqueness without growth condition for the mean curvature equation with radial initial data

Samuel Biton, Emmanuel Chasseigne and Olivier Ley

Laboratoire de Mathématiques et de Physique Théorique

Université de Tours, Parc de Grandmont, 37200 Tours, France

ABSTRACT. We prove the uniqueness of a solution to the mean curvature equation for graphs

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2 u Du, Du \rangle}{1 + |Du|^2} = 0 \quad \text{in } I\!R^N \times (0, +\infty),$$

for any radial continuous initial data $u(x,0) = u_0(|x|)$. The existence of a smooth solution to this problem comes from the work of Ecker and Huisken [18]. We obtain in addition that the solution is radial. We point out that existence and uniqueness hold without any growth restriction on the solution or the initial data, a situation which is rather different to the related stationary problem: in this case, we show there is a limiting growth above which uniqueness does not hold anymore. An application of the uniqueness result to the evolution by mean curvature of entire radial graphs is given.

Keywords and Phrases. Mean curvature equation, Quasilinear equations, Uniqueness without growth restriction, Viscosity solutions, Level-set approach, Mean curvature flow.

AMS Subject Classification. 35K55, 35B05, 35K65, 53C44, 35J70.

1 Introduction

In [18], Ecker and Huisken proved the existence of a classical solution $u \in C^\infty(I\!R^N \times (0,+\infty)) \cap C(I\!R^N \times [0,+\infty))$ of the mean curvature equation for graphs

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2 u Du, Du\rangle}{1+|Du|^2} = 0 \quad \text{in } I\!R^N \times (0,+\infty), \tag{1.1}$$

for any initial data $u_0 \in W^{1,\infty}_{loc}(I\!R^N)$. This existence result, which can be extended to merely continuous initial data [6], holds without any growth condition on u_0, which appears to be rather surprising when comparing to the case of the heat equation (where exponential growth of the initial data has to be required [1]).

Our aim is to study the uniqueness of the solution Ecker and Huisken construct. This question has been already addressed in some situations: in [6], uniqueness for convex initial data is shown without any growth condition. This result is extended in [9] for "convex at infinity" initial data. The case of dimension 1 was completely solved independently by Chou and Kwong [13] and in [5]. A result in arbitrary dimension is given in [4] under growth conditions on the gradient of the initial data. In this paper, we prove the result when u_0 is radial:

Theorem 1.1 *Let $u_0 \in C(I\!R^N)$ be radial. Then there exists a unique continuous viscosity solution u to (1.1) with initial data u_0. Moreover, $u \in C^\infty(I\!R^N \times (0,+\infty)) \cap C(I\!R^N \times [0,+\infty))$ and $u(\cdot,t)$ is radial for any $t \geqslant 0$.*

The general case is still open and Theorem 1.1 seems to be the first positive result in dimension $N > 1$ for solutions which can oscillate arbitrarily at infinity. As we said above, this situation, *i.e.* existence and uniqueness without growth condition, is quite unusual. Let us mention that, to the best of our knowledge, apart from equations of type (1.1), we know only a few cases where similar features happen for parabolic pdes: for first-order Hamilton-Jacobi equations for which one has a "finite speed of propagation" property (see [16], [24], [26]). The same kind of results appears in presence of absorption [10], thanks to the localization properties of the equation. But closer to the present situation is the so-called fast-diffusion equation $u_t = \Delta(u^m)$, $0 < m < 1$ [22, 11], which is purely diffusive and nevertheless offers existence without any growth restriction.

On the other hand, this context is rather different than the stationary one. In Section 6, we develop the example of a stationary equation related to (1.1): we prove uniqueness under exponential-type growth conditions for the solutions and give an example of non-uniqueness when this limiting growth is not satisfied.

One of the motivations of this work is related to the generalized evolution by mean curvature of hypersurfaces. The link appears when noticing that, for any $t \geqslant 0$, the set Graph$(u(\cdot,t))$ of any solution u of (1.1) can be seen as an hypersurface of $I\!R^{N+1}$ evolving in time with a normal speed equal to the mean curvature. Now, it turns out from [6] that the uniqueness result for (1.1) is equivalent to the non fattening of the generalized evolution of Graph(u_0) defined via the level set approach (see Osher and Sethian [28], Chen, Giga and Goto [12], Evans and Spruck [19] or [6] and Section 2 for a description of this approach). Therefore, one by-product of Theorem 1.1 is the non-fattening of the generalized evolution by mean curvature when starting with an hypersurface which is a continuous radial entire graph (see Section 7).

Let us now describe our strategy to prove Theorem 1.1. First, we use the level-set approach to prove that, for any radial initial data $u_0 \in C(\mathbb{R}^N)$, the maximal and minimal solutions to (1.1), noted u^+ and u^- respectively, are radial. Moreover, we show that uniqueness for C^1 initial data implies also uniqueness for continuous initial data, a simplification we use to restrict ourselves to regular data.

Then, a standard way to get uniqueness is to apply the maximum principle on $u^+ - u^-$, but to do so, one needs to control the behaviour of $u^+ - u^-$ as $|x| \to \infty$. The first step in this direction is provided by the following integral estimate: we denote by $\varphi^\pm$ the functions defined by $u^\pm(x,t) = \varphi^\pm(|x|,t)$. Then for any $T > 0$ and $r_0 \geqslant 1$, there exists $C = C(T)$ such that

$$(1.2) \qquad \int_{r_0}^{+\infty} (\varphi^+ - \varphi^-)(r,t)dt \leqslant C \quad \text{for any } t \in [0,T].$$

Then, we construct smooth, radial solutions with extremal gradients. More precisely, if $u(x,t) = \varphi(|x|,t)$ is a radial solution of (1.1), the function φ is a solution of

$$(1.3) \qquad \partial_t \varphi - \frac{\partial^2_{rr}\varphi}{1+(\partial_r\varphi)^2} - (N-1)\frac{\partial_r\varphi}{r} = 0 \quad \text{in } (0,+\infty)\times(0,+\infty),$$

and we prove that there exists $u^{g,+}(x,t) = \varphi^{g,+}(|x|,t)$ and $u^{g,-} = \varphi^{g,-}(|x|,t)$ such that, for any other smooth radial solution $u(x,t) = \varphi(|x|,t)$, there holds

$$\partial_r \varphi^{g,-} \leqslant \partial_r \varphi \leqslant \partial_r \varphi^{g,+} \quad \text{in } [0,+\infty)\times(0,+\infty).$$

The proof of this result relies upon a comparison argument for the derived equation satisfied by $\psi = \partial_r \varphi$, namely for any $R > 0$

$$(1.4) \qquad \partial_t \psi - \partial^2_{rr}(\arctan\psi) - (N-1)\partial_r\left(\frac{\psi}{r}\right) = 0 \quad \text{in } (0,R)\times(0,T).$$

The comparison for (1.4) is obtained by using the so-called "dual method", through the distributional formulation of the equation.

Then, using (1.2), simple arguments show that

$$u^+ = u^{g,-} \quad \text{and} \quad u^- = u^{g,+},$$

so that $u^+ - u^-$ is nonincreasing. The last step consists in applying the standard maximum principle in $[0,+\infty)\times[0,T]$ for any $T > 0$ and conclude that

$$u^+ \equiv u^-.$$

The paper is organized as follows: in Section 2, we recall the level set approach applied to the evolution by mean curvature of entire graphs in $\mathbb{R}^{N+1}$ and we construct a minimal and a maximal solution to (1.1), which are proved to be radial and smooth. We prove the integral estimate in Section 3. Section 4 is devoted to the construction of the extremal gradient solutions: we prove first a comparison result for (1.4) on bounded sets and then study a Neumann problem with radial symmetry which are used in the construction. In Section 5, we conclude with the proof of Theorem 1.1. We add some remarks concerning the stationary problem in Section 6 and conclude with an application to the mean curvature flow in Section 7.

Acknowledgment. We are pleased to thank Guy Barles for fruitful discussions.

2 The geometrical approach. Extremal Solutions

Theorem 2.1 *Let $u_0 \in C(\mathbb{R}^N)$ be radial. Then (1.1) has a maximal (respectively minimal) solution $u^+ \in C^\infty(\mathbb{R}^N \times (0,+\infty)) \cap C(\mathbb{R}^N \times [0,+\infty))$ (respectively $u^- \in C^\infty(\mathbb{R}^N \times (0,+\infty)) \cap C(\mathbb{R}^N \times [0,+\infty))$) which is radial in x for any $t \geqslant 0$.*

This theorem is a consequence of Theorem 8.1 of [6]. Before giving the proof of Theorem 2.1, we need to recall some facts about the level set method used in [6] to study quasilinear equations like (1.1) (we refer to this paper for further details).

First, we introduce some notations. In the canonical basis of $\mathbb{R}^{N+1}$, we write any point z in the form $z = (x, y)$ where $x = (z_1, z_2, \cdots, z_N) \in \mathbb{R}^N$ and $y = z_{N+1} \in \mathbb{R}$. By the axis (Oy), we mean the vectorial line spanned by the last vector of the basis.

The geometrical approach consists in seeing the graph of any solution of (1.1) as an hypersurface evolving (in time) in $\mathbb{R}^{N+1}$. The geometrical evolution associated with (1.1) is the motion by mean curvature (see Section 7). An alternative way to describe this motion is to use the level-set method developed in [28], [19] and [12]. The level-set method applies to the motion of general hypersurfaces $\Gamma_0 \subset \mathbb{R}^{N+1}$; here, for simplicity, we restrict ourselves to the motion of graphs. Let u a solution of (1.1) with initial data u_0. Set $\Gamma_0 = \mathrm{Graph}(u_0)$ and $\Omega_0^+ = \{(x,y) \in \mathbb{R}^{N+1} : y > u_0(x)\}$. We consider a uniformly continuous (UC for short) function $v_0 : \mathbb{R}^{N+1} \to \mathbb{R}$ such that

$$(2.1) \qquad \{v_0 = 0\} = \Gamma_0, \quad \{v_0 > 0\} = \Omega_0^+ \quad \text{and} \quad \{v_0 < 0\} = \mathbb{R}^{N+1} - (\Omega_0^+ \cup \Gamma_0).$$

Then, we are looking for a function $v : \mathbb{R}^{N+1} \times [0, +\infty) \to \mathbb{R}$ such that, for every $(x,t) \in \mathbb{R}^N \times (0, +\infty)$ and every solution u of (1.1),

$$v(x, u(x,t), t) = 0 \quad \text{and} \quad v(x, u_0(x), 0) = 0.$$

It follows that v has to solve the (geometrical) mean curvature equation

$$(2.2) \qquad \frac{\partial v}{\partial t} - \Delta v + \frac{\langle D^2 v Dv, Dv \rangle}{|Dv|^2} = 0 \quad \text{in } \mathbb{R}^{N+1} \times (0, +\infty),$$

with initial data $v(z,0)=v_0(z)$ for any $z\in \mathbb{R}^{N+1}$.

Thanks to the theory of viscosity solutions, (2.2) is well-posed in the class of uniformly continuous functions : if $v_0\in UC(\mathbb{R}^{N+1})$, then there exists a unique viscosity solution $v\in UC(\mathbb{R}^{N+1}\times[0,+\infty)$ (for viscosity solutions, we refer to Crandall, Ishii and Lions [14]).

Moreover, the fundamental result of the level set method states that the 0-level set of $v(\cdot,t)$ at each time t depends only on (Γ_0,Ω_0^+) and (2.2), but not on the functions v_0 and v. This allows to define the generalized motion by mean curvature of (Γ_0,Ω_0^+) by the family $(\Gamma_t,\Omega_t^+)_{t\geqslant 0}$, where

$$\Gamma_t := \{v(\cdot,t)=0\},\quad \Omega_t^+ := \{v(\cdot,t)>0\}\quad \text{for all } t\geqslant 0.$$

The set $\displaystyle\bigcup_{t\geqslant 0}\Gamma_t\times\{t\}$ is called the front and Γ_t the front at time t.

The evolution is generalized in the sense that the front may develop singularities or interior (the front is said to "fatten" in this last case) in $\mathbb{R}^{N+1}\times[0,+\infty)$. This fattening phenomenon is equivalent to the nonuniqueness of solutions of (1.1) for the initial data u_0; we refer to [6] for proofs and precise results.

Proof of Theorem 2.1. The existence extremal solutions u^+ and u^- is given by [6, Theorem 8.1] : for any $(x,t)\in\mathbb{R}^N\times(0,+\infty)$,

$$u^+(x,t)=\sup\{y\in\mathbb{R}:(x,y)\in\Gamma_t\}\quad\text{and}\quad u^-(x,t)=\inf\{y\in\mathbb{R}:(x,y)\in\Gamma_t\}$$

(roughly speaking, the graph of u^+ is the upper-boundary of the front and the graph of u^- is the lower-boundary). The regularity is proved in [6, Theorem 9.1]. It remains to show that u^+ and u^- are radial. It is a consequence of the following lemma the proof of which is postponed.

Lemma 2.2 *Let Ω_0^+ be an open subset of $\mathbb{R}^{N+1}$ with boundary $\Gamma_0=\partial\Omega_0^+$ and $A\in\mathcal{O}_{N+1}$, where $\mathcal{O}_{N+1}$ is the group of isometries of $\mathbb{R}^{N+1}$. Consider the generalized evolution by mean curvature $(\Gamma_t)_{t\geqslant 0}$ (respectively $(\tilde\Gamma_t)_{t\geqslant 0}$) of Γ_0 (respectively $A(\Gamma_0)$). Then*

$$\tilde\Gamma_t = A(\Gamma_t)\quad \textit{for all } t\geqslant 0.$$

This lemma means that the generalized evolution by mean curvature commutes with isometries.

Now, if u_0 is radial, then $\Gamma_0:=\text{Graph}(u_0)$ is invariant by any rotation A of axis (Oy). Let Γ_t and $\tilde\Gamma_t$ be the generalized evolutions of Γ_0 and $A(\Gamma_0)$ respectively. On the one hand, $\tilde\Gamma_t=\Gamma_t$ since $A(\Gamma_0)=\Gamma_0$, and one the other hand $\tilde\Gamma_t=A(\Gamma_t)$ by the lemma. It follows that Γ_t is invariant by A. Therefore $u^+(\cdot,t)$ and $u^-(\cdot,t)$ are radial. ■

Proof of Lemma 2.2. We use the level set method : let $v_0\in UC(\mathbb{R}^{N+1})$ such that (2.1) holds and let v be the unique uniformly continuous viscosity solution of (2.2) with initial data v_0; then, by definition, $\Gamma_t=\{v(\cdot,t)=0\}$.

Let $\tilde{v}_0(z) := v_0(A^T z)$ for any $z \in \mathbb{R}^{N+1}$. Note that

$$\tilde{v}_0(x,y) = 0 \Leftrightarrow (x,y) \in A(\Gamma_0) \quad \text{and} \quad \tilde{v}_0(x,y) > 0 \Leftrightarrow (x,y) \in A(\Omega_0^+).$$

It follows that $\tilde{\Gamma}_t = \{\tilde{v}(\cdot,t) = 0\}$, where $\tilde{v}$ is the unique UC viscosity solution of (2.2) with initial data $\tilde{v}_0$.

Let $w(z,t) = v(A^T z, t)$ for every $(z,t) \in \mathbb{R}^{N+1} \times [0,+\infty)$. A straightforward computation gives

$$\frac{\partial v}{\partial t} = \frac{\partial w}{\partial t}, \quad Dv = A^T Dw \quad \text{and} \quad D^2 v = A^T D^2 w A.$$

Since v is a solution of (2.2), it follows that $\tilde{v}$ is a solution of

$$\frac{\partial w}{\partial t} - \operatorname{Tr}\left[A\left(I - \frac{(A^T Dw)\otimes(A^T Dw)}{|A^T Dw|^2}\right) A^T D^2 w\right] = 0 \quad \text{in } \mathbb{R}^{N+1} \times (0,+\infty).$$

For any $\xi \in \mathbb{R}^{N+1}$, since A is an isometry, we have

$$\begin{aligned}\langle A(A^T Dw)\otimes(A^T Dw)A^T\xi, \xi\rangle &= \langle (A^T Dw)\otimes(A^T Dw)A^T\xi, A^T\xi\rangle = \langle A^T Dw, A^T\xi\rangle^2 \\ &= \langle Dw, \xi\rangle^2 = \langle (Dw\otimes Dw)\xi, \xi\rangle.\end{aligned}$$

It proves that w is a solution of (2.2) with initial data $w(z,0) = v_0(A^T z) = \tilde{v}_0$. By uniqueness for (2.2) in UC, we obtain that $w = \tilde{v}$; thus, $\tilde{\Gamma}_t = \{w(\cdot,t) = 0\} = \{z \in \mathbb{R}^{N+1} : v(A^T z, t) = 0\} = A(\Gamma_t)$, which completes the proof of the lemma. ■

We conclude this section by a result showing that we can restrict ourselves to C^1 radial initial data in the proof of Theorem 1.1.

Proposition 2.3 *Suppose that Theorem 1.1 holds for radial initial data $u_0 \in C^1(\mathbb{R}^N)$. Then, it holds also for radial initial data $u_0 \in C(\mathbb{R}^N)$.*

Proof of Proposition 2.3. Let $u_0 \in C(\mathbb{R}^N)$ be radial and, for any $\varepsilon > 0$, let $u_0^\varepsilon \in C^1(\mathbb{R}^N)$ such that $\|u_0 - u_0^\varepsilon\|_{\infty,\mathbb{R}^N} \leqslant \varepsilon/2$. Note that we can choose u_0^ε to be a radial function. We define $\overline{u}_0^\varepsilon = u_0^\varepsilon + \varepsilon/2$ and $\underline{u}_0^\varepsilon = u_0^\varepsilon - \varepsilon/2$. Then

$$\underline{u}_0^\varepsilon \leqslant u_0 \leqslant \overline{u}_0^\varepsilon \quad \text{and} \quad \|\overline{u}_0^\varepsilon - \underline{u}_0^\varepsilon\|_{\infty,\mathbb{R}^N} \leqslant \varepsilon.$$

Let u (respectively $\overline{u}^\varepsilon$, $\underline{u}^\varepsilon$) be a solution of (1.1) with initial data u_0 (respectively $\overline{u}_0^\varepsilon$, $\underline{u}_0^\varepsilon$).

Since the equation (1.1) depends only on the derivatives of the solution and since, by assumption, we have uniqueness for initial datas $\overline{u}_0^\varepsilon$ and $\underline{u}_0^\varepsilon$, we obtain

$$\overline{u}^\varepsilon = u^\varepsilon + \frac{\varepsilon}{2} \quad \text{and} \quad \underline{u}^\varepsilon = u^\varepsilon - \frac{\varepsilon}{2}.$$

Thus

$$\|\overline{u}^\varepsilon - \underline{u}^\varepsilon\|_{\infty,\mathbb{R}^N\times[0,+\infty)} \leqslant \varepsilon. \tag{2.3}$$

But, since uniqueness holds for $\underline{u}^\varepsilon$ and $\overline{u}^\varepsilon$ and $\underline{u}_0^\varepsilon \leqslant u_0 \leqslant \overline{u}_0^\varepsilon$, it follows from the geometrical approach (see [6]) that the generalized evolution Γ_t of Graph(u_0) is such that

$$\Gamma_t \subset \{(x,y) \in I\!\!R^{N+1} : \underline{u}^\varepsilon(x,t) \leqslant y \leqslant \overline{u}^\varepsilon(x,t)\}.$$

Therefore we obtain

$$\|u^+ - u^-\|_{\infty, I\!\!R^N \times [0,+\infty)} \leqslant \varepsilon$$

and we get the result letting ε go to 0^+. ■

Remark 2.4 In the proof of the previous proposition, we do not use that u_0 is radial. The result is more general: uniqueness for (1.1) for continuous initial datas holds as soon as we can prove it for smooth initial datas.

3 An integral estimate

Let us first introduce some notations: we define $\varphi^\pm$ by $u^\pm(x,t) = \varphi^\pm(|x|,t)$ and set, for $r_0 \geqslant 1$ and every $(r,t) \in [r_0, +\infty) \times [0, +\infty)$,

$$\psi^\pm(r,t) = \int_{r_0}^r \varphi^\pm(\rho,t)d\rho \quad \text{and} \quad \psi = \psi^+ - \psi^-.$$

Then the following integral estimate holds:

Lemma 3.1 *Let $T > 0$. There exists a positive constant $C = C(T)$ such that, for any $r \geqslant r_0$ and $t \in [0,T]$, we have*

$$\psi(r,t) = \int_{r_0}^r (\varphi^+ - \varphi^-)(\rho,t)d\rho \leqslant C.$$

Proof of Lemma 3.1. Since φ^+ and φ^- are solutions of (1.3), by integrating the equation, we get that ψ^+ and ψ^- are solutions of

$$\partial_t \phi - [\arctan \partial_r \phi(\rho,t)]_{r_0}^r - (N-1)\int_{r_0}^r \frac{\partial_{rr}^2 \phi(\rho,t)}{\rho} d\rho = 0.$$

We integrate by parts and obtain

$$\partial_t \phi - [\arctan \partial_r \phi(\rho,t)]_{r_0}^r - (N-1)\left[\frac{\partial_r \phi(\rho,t)}{\rho}\right]_{r_0}^r - (N-1)\int_{r_0}^r \frac{\partial_r \phi(\rho,t)}{\rho^2} d\rho = 0.$$

Subtracting the equalities for ψ^+ and ψ^-, one gets that ψ satisfies

$$\partial_t \psi - (N-1)\left[\frac{\partial_r \psi(\rho,t)}{\rho}\right]_{r_0}^r - (N-1)\int_{r_0}^r \frac{\partial_r \psi(\rho,t)}{\rho^2} d\rho \leqslant \pi.$$

Therefore, using $\partial_r \psi = \varphi^+ - \varphi^- \geqslant 0$ and $\rho^2 \geqslant r_0^2 \geqslant 1$ in the last integral, we have for any $r \geqslant r_0$,

$$\partial_t \psi - (N-1)\frac{\partial_r \psi}{r} - (N-1)\psi \leqslant \pi.$$

Changing ψ in $\bar{\psi} = \mathrm{e}^{-(N-1)t}\psi$, we obtain that $\bar{\psi}$ satisfies

$$\partial_t \bar{\psi} - (N-1)\frac{\partial_r \bar{\psi}}{r} \leqslant \pi \quad \text{in } [r_0, +\infty) \times [0, +\infty).$$

Now, for fixed $T > 0$, we use a "friendly giant" to get a bound on $\bar{\psi}$. Let $R \geqslant 3 + (N-1)T$, $C = C(T) = \sup_{[0,T]} |\bar{\psi}(r_0, \cdot)|$, and consider

$$W(r,t) = \frac{1}{R - (N-1)t - r} + \pi \mathrm{e}^{2t} + C.$$

Straightforward computations show that $\partial_t W - (N-1)(\partial_r W)/r > \pi$, while the boundary conditions are ordered: $\bar{\psi}(\cdot, 0) = 0 \leqslant W(\cdot, 0)$, $\bar{\psi}(r_0, \cdot) \leqslant W(r_0, \cdot)$ and for any $t \in [0,T], W(r,t) \to +\infty$ as $r \to R - (N-1)t$. It follows that

$$\bar{\psi} \leqslant W \quad \text{in} \quad \big\{(r,t) \in [r_0, +\infty) \times [0,T] : R - (N-1)t - r > 0\big\}.$$

Letting $R \to +\infty$, we obtain $\bar{\psi} \leqslant \pi \mathrm{e}^{2T} + C$ in $[r_0, +\infty) \times [0, T)$, and finally

$$\psi(r,t) \leqslant (\pi \mathrm{e}^{2T} + C(T))\mathrm{e}^{(N-1)T},$$

which ends the proof. ■

4 Construction of solutions with extremal gradients

In this section, we construct solutions to (1.1) with extremal gradients. More precisely, we prove

Theorem 4.1 *For any radial initial data $u_0 \in C^1(\mathbb{R}^N)$, there exist two smooth radial solutions of (1.1), $u^{g,+}(x,t) = \varphi^{g,+}(|x|,t)$ and $u^{g,-}(x,t) = \varphi^{g,-}(|x|,t)$ such that, for any smooth radial solution $u(x,t) := \varphi(|x|,t)$, we have*

$$\partial_r \varphi^{g,-}(r,t) \leqslant \partial_r \varphi(r,t) \leqslant \partial_r \varphi^{g,+}(r,t) \quad \textit{for every } (r,t) \in [0,+\infty) \times [0,+\infty).$$

Before giving the proof of this theorem, we establish a comparison theorem for (1.4) in bounded sets and then study a preliminary Neumann problem.

4.1 A comparison result for the derived equation

We state below a comparison result for a slightly more general equation than the derived equation (1.4). More precisely, for $\varepsilon \geqslant 0$, we consider

$$(4.1) \qquad P_\varepsilon \equiv \partial_t \psi - \varepsilon \partial^2_{rr} \psi - \partial_{rr}(\arctan \psi) - (N-1)\partial_r \left(\frac{\psi}{r}\right) = 0$$

in $(0,+\infty) \times (0,+\infty)$. The method we employ has been extensively used by a number of authors in diffusion equations (for instance Aronson, Crandall and Peletier [2], Dahlberg and Kenig [17], etc.). It relies upon resolution of the dual equation.

Proposition 4.2 *Let $\varepsilon \geqslant 0$, $R, T > 0$ and $\psi_1, \psi_2 \in C^2((0,R)\times(0,T)) \cap C([0,R]\times[0,T])$ be respectively classical sub- and supersolution of (4.1) in $(0,R)\times(0,T)$ such that*

$$\begin{aligned} \psi_1(\cdot,0) \leqslant \psi_2(\cdot,0) & \quad \textit{on} \quad [0,R], \\ \psi_1(t,R) \leqslant \psi_2(t,R) & \quad \textit{for} \quad t\in[0,T], \\ \psi_1(t,0) = \psi_2(t,0) = 0 & \quad \textit{for} \quad t \in [0,T]. \end{aligned}$$

Then $\psi_1 \leqslant \psi_2$ on $[0,R]\times[0,T]$.

Proof of Proposition 4.2. From the definition of ψ_1 and ψ_2 we have,

$$P_\varepsilon(\psi_1) \leqslant 0 \leqslant P_\varepsilon(\psi_2) \quad \text{on} \quad (0,R)\times(0,T).$$

Then for $\rho > 0$, and $0 < \tau < T$, let us consider a nonnegative function $\chi \in \mathcal{C}^2([\rho, R-\rho]\times[0,\tau])$, so that

$$\int_0^\tau \int_\rho^{R-\rho} \big(P_\varepsilon(\psi_1) - P_\varepsilon(\psi_2)\big)(r,t)\chi(r,t)drdt \leqslant 0.$$

Integrating by parts on $[0,\tau]\times[\rho,R-\rho]$, we obtain

$$\text{(4.2)} \quad \int_\rho^{R-\rho} (\psi_1-\psi_2)\chi(r,\tau)dr \leqslant$$
$$\int_0^\tau \int_\rho^{R-\rho} (\psi_1-\psi_2)\left(\partial_t\chi + A_\varepsilon(r,t)\partial_{rr}\chi - \frac{(N-1)(1+\varepsilon)\partial_r\chi}{r}\right)drdt + \mathcal{B}_\varepsilon(\rho,\tau),$$

where $A_\varepsilon(r,t)$ is defined by

$$A_\varepsilon(r,t) = \varepsilon + \begin{cases} \dfrac{\arctan\psi_1 - \arctan\psi_2}{\psi_1-\psi_2} & \text{if} \quad \psi_1 \neq \psi_2 \\ \dfrac{1}{1+(\psi_1)^2} & \text{if} \quad \psi_1 = \psi_2, \end{cases}$$

and $\mathcal{B}_\varepsilon(\rho,\tau)$ contains the various boundary terms due to the integration by parts :

$$\mathcal{B}_\varepsilon(\rho,\tau) = \int_\rho^{R-\rho} [(\psi_1-\psi_2)\chi(r,0)]dr - \int_0^\tau [(\arctan\psi_1 - \arctan\psi_2)\cdot\partial_r\chi]_\rho^{R-\rho}\,dt$$
$$+ \int_0^\tau [\partial_r(\arctan\psi_1 - \arctan\psi_2)\cdot\chi]_\rho^{R-\rho}\,dt - (N-1)\int_0^\tau \left[\frac{\psi_1-\psi_2}{r}\chi\right]_\rho^{R-\rho} dt.$$

Note that

$$\begin{aligned} 1+\varepsilon \geqslant A_\varepsilon(r,t) &= \varepsilon + \int_0^1 (\arctan)'(\lambda\psi_1(r,t) + (1-\lambda)\psi_2(r,t))\,d\lambda \\ &= \varepsilon + \int_0^1 \frac{d\lambda}{1+(\lambda\psi_1(r,t)+(1-\lambda)\psi_2(r,t))^2} \\ &\geqslant \varepsilon + \inf_{[\rho,R-\rho]\times[0,\tau]} \left\{\frac{1}{1+\psi_1^2}, \frac{1}{1+\psi_2^2}\right\} \geqslant \varepsilon + \mu, \end{aligned}$$

where $\mu = \mu(R) > 0$. Moreover, the coefficient A_ε is as regular as ψ_1 and ψ_2 are.

We will now solve the dual backward equation on χ which appears in (4.2), with special boundary data in order to control $\mathcal{B}_\varepsilon$. We thus need the following lemma which is a consequence of classical results on linear parabolic equation (see for example Friedman [20], Ladyzenskaja et al. [25] or Lieberman [27]).

Lemma 4.3 *For any fixed nonnegative function* $\theta \in C_0([0,R])$ *and any* $\rho > 0$ *such that* $\operatorname{supp}(\theta) \subset (\rho, R-\rho)$ *there exists a nonnegative function* $\chi = \chi_{\varepsilon,\rho} \in C^2((\rho, R-\rho) \times (0,\tau)) \cap C^1([\rho, R-\rho] \times [0,\tau])$ *which is a solution of the linear backward equation*

$$\partial_t \chi + A_\varepsilon(r,t)\partial_{rr}\chi - (N-1)\frac{(1+\varepsilon)\partial_r \chi}{r} = 0 \quad on \quad (\rho, R-\rho) \times (0,\tau)$$

satisfying the boundary conditions:

$$\begin{cases} \chi(\cdot,\tau) = \theta & on \quad [\rho, R-\rho], \\ \chi(R-\rho,\cdot) = \chi(\rho,\cdot) = 0 & on \quad [0,\tau]. \end{cases}$$

Moreover there exists a positive constant $C(R,\theta)$ *(independent of* ε*) such that*

$$-C(R,\theta) \leqslant \partial_r \chi(R-\rho,t) \leqslant 0 \leqslant \partial_r \chi(\rho,t) \leqslant C(R,\theta) \quad \text{for} \quad t \in [0,\tau]. \tag{4.3}$$

We postpone the proof of this lemma and conclude the proof of the proposition. Note that for such $\chi = \chi_{\varepsilon,\rho}$, we have

$$\int_\rho^{R-\rho} (\psi_1 - \psi_2)\chi(r,\tau)dr \leqslant \mathcal{B}_\varepsilon(\rho,\tau).$$

Since $\chi = 0$ at $r = \rho$ and $r = R-\rho$, the remaining boundary terms are the following:

$$\mathcal{B}_\varepsilon(\rho,\tau) = \int_\rho^{R-\rho} [(\psi_1 - \psi_2)\chi(r,0)]dr + \int_0^\tau \left[(\arctan\psi_1 - \arctan\psi_2)(\cdot,t)\partial_r\chi(\cdot,t)\right]_\rho^{R-\rho} dt.$$

We remark that $\psi_1(x,0) \leqslant \psi_2(x,0)$ while $\chi \geqslant 0$, so that the boundary term at time $t = 0$ is nonpositive. Then we use the bounds on $\partial_r\chi$ at $r = \rho$ and $r = R-\rho$, which give

$$\mathcal{B}_\varepsilon(\rho,\tau) \leqslant C(R,\theta)\left[\int_0^\tau (\arctan\psi_1 - \arctan\psi_2)^+(\rho,t)dt + \int_0^\tau (\arctan\psi_1 - \arctan\psi_2)^+(R-\rho,t)dt\right],$$

where the notation f^+ stands here for $\max\{f,0\}$. Since $\chi(r,\tau) = \theta(r)$, we get the estimate

$$\int_\rho^{R-\rho} (\psi_1 - \psi_2)(r,\tau)\theta(r)dr \leqslant C(R,\theta)\left[\max_{(0,\tau)}(\psi_1 - \psi_2)^+(\rho,t) + \max_{(0,\tau)}(\psi_1 - \psi_2)^+(R-\rho,t)\right].$$

Now we let ρ decrease to zero, remembering that $\psi_1(0,t) = \psi_2(0,t) = 0$, and $\psi_1(R,\cdot) \leqslant \psi_2(R,\cdot)$, so that by continuity of both solutions up to the boundary, we get

$$\int_0^R (\psi_1 - \psi_2)(r,\tau)\theta(r)dr \leqslant 0,$$

for any $\theta \in C_0([0,R])$ as above.

By continuity of the functions it implies that $\psi_1(\cdot,\tau) \leqslant \psi_2(\cdot,\tau)$ on $[0,R]$. This is the desired result since τ is arbitrary in the previous reasoning, which ends the proof of Proposition 4.2. ■

We turn to the proof of the lemma

Proof of Lemma 4.3. From the definition of A_ε, we know that $A_\varepsilon \in C^\infty([\rho, R-\rho] \times [0,\tau])$ and that

$$\inf_{[0,R]\times[0,\tau]} A_\varepsilon(r,t) \geqslant \mu(R) > 0.$$

Therefore, classical results on linear parabolic equation (see [20], [25] or [27] for example) provide the existence of a classical solution χ to the Cauchy-Dirichlet problem of Lemma 4.3; because of the maximum principle, we have $0 \leqslant \chi \leqslant 1$ on $[\rho, R-\rho] \times [0,\tau]$.

It remains to prove the gradient estimates (4.3) using some barrier arguments. We start with $\partial_r\chi(R-\rho,\cdot)$. Let $w(r,t) = C(R-\rho-r)(1-\tilde{C}(R-\rho-r))$ for $(r,t) \in [\rho, R-\rho] \times [0,\tau]$, where

$$\tilde{C} = \tilde{C}(R) = \frac{2(N-1)}{R\mu(R)}, \quad C = C(R,\theta) = 2\max\Big\{2\tilde{C}+1, \sup_{[\rho,R-\rho]} |\partial_r\theta|\Big\}. \tag{4.4}$$

We claim that w is a strict subsolution of the problem of Lemma 4.3 in $[R_0, R-\rho] \times [0,\tau]$, where $R_0 = R-\rho-1/(2\tilde{C})$. Indeed,

$$\partial_t w = 0, \quad \partial_r w = C(-1+2\tilde{C}(R-\rho-r)), \quad \partial_{rr} w = -2C\tilde{C},$$

hence

$$\partial_t w + A_\varepsilon w_{rr} - (N-1)(1+\varepsilon)\frac{\partial_r w}{r} \leqslant -2\mu(R)C\tilde{C} + (N-1)(1+\varepsilon)\frac{C}{r} < 0,$$

since we can assume $\varepsilon \leqslant 1$ and $R_0 > R/2$.

Now, we check the boundary conditions:

— On $[R_0, R-\rho] \times \{t=\tau\}$. By the mean value theorem, for any $r \in [R_0, R-\rho]$,

$$\theta(r) - \theta(R-\rho) \leqslant \sup_{[\rho,R-\rho]} |\partial_r\theta|(R-\rho-r).$$

Using $\theta(R-\rho) = 0$ and (4.4), we obtain

$$\theta(r) \leqslant \frac{1}{2}C(R,\theta)(R-\rho-r) \leqslant C(R,\theta)(R-\rho-r)(1-\frac{1}{2}).$$

But, since $r \geqslant R_0 = R - \rho - 1/(2\tilde{C})$, we get

$$\tilde{C}(R-\rho-r) \leqslant \frac{1}{2};$$

thus $\chi(r,t) = \theta(r) \leqslant w(r,t)$ on $[R_0, R-\rho] \times \{t=\tau\}$.
— On $\{r = R-\rho\} \times [0,\tau]$, $w(R-\rho,t) = 0 \geqslant \chi(R-\rho,t)$.
— On $\{r = R_0\} \times [0,\tau]$. We have

$$w(R_0,t) = \frac{C}{2\tilde{C}} \geqslant \frac{2\tilde{C}+1}{2\tilde{C}} \geqslant 1 \geqslant \chi(R_0,t),$$

so that the claim is proved.

Applying the maximum principle, we get $\chi(r,t) \leqslant w(r,t)$ in $[R_0, R-\rho] \times [0,\tau]$, and it follows

$$0 \geqslant \chi(R-\rho,t) - \chi(r,t) \geqslant w(R-\rho,t) - w(r,t).$$

Dividing by $R-\rho-r$ and letting $r \to R-\rho$, we obtain the desired gradient bound,

$$0 \geqslant \partial_r\chi(R-\rho,t) \geqslant -C(R,\theta), \quad \text{for } t \in [0,\tau].$$

The proof of the estimate for $\partial_r\chi(\rho,\cdot)$ is much simpler. It is sufficient to consider $w(r,t) = (\sup|\partial_r\theta|+1)(r-\rho)$ which is a strict subsolution in $[\rho, R-\rho] \times [0,\tau]$. We have

$$C(R,\theta)(r-\rho) \geqslant (\sup|\partial_r\theta|+1)(r-\rho) \geqslant \chi(r,t) \geqslant 0 \quad \text{in } [\rho, R-\rho] \times [0,\tau],$$

which gives the conclusion dividing by $r-\rho$ and letting $r \to \rho$. It completes the proof of the lemma. ■

4.2 A Neumann problem

In this section, we study equation (1.1) in a bounded domain, with Neumann conditions, which is needed to construct extremal gradient solutions in the following section. In the sequel, $R > 1$ is a real number and $B_R(0)$ denotes the ball of radius R centered at $x = 0$. For the sake of notation, we denote the diffusion matrix of (1.1) by

$$b(p) = I - \frac{p \otimes p}{1+|p|^2} \quad \text{for } p \in \mathbb{R}^N.$$

Proposition 4.4 *Let $u_0 \in C^1(\overline{B_R(0)})$ be a radial initial data, $T > 0$ and $K \in \mathbb{R}$. Then there exists a radial solution $u \in C^2(B_R(0) \times (0,T)) \cap C^1(\overline{B_R(0)} \times [0,T])$ to the following problem:*

$$(4.5) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{Tr}(b(Du)D^2u) = 0 & \text{in } B_R(0) \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu}(x,t) = K & \text{on } \{|x| = R\} \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } B_R(0) \times \{t=0\}. \end{cases}$$

Remark 4.5 Of course the above problem requires a compatibility of the data at the boundary $\{|x| = R\} \times \{t = 0\}$ to be clearly defined, but it is easy to modify a little bit u_0 near the boundary in order to ensure this compatibility. We skip this detail for the sake of clarity. This point is not important for our purpose since our aim is to send R to $+\infty$ later on.

Proof of Proposition 4.4. The proof of this proposition will follow from the standard "vanishing viscosity method". We introduce an approximate problem, show some estimates for the solutions and finally extract a subsequence which converges to a smooth solution of the original problem.

Step 1. The approximate problem.
For any $\varepsilon > 0$, we consider:

$$
(4.6) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \varepsilon \Delta u - \operatorname{Tr}(b(Du)D^2 u) = 0 & \text{in } B_R(0) \times (0, +\infty), \\ \dfrac{\partial u}{\partial \nu}(x,t) = K & \text{on } \{|x| = R\} \times (0, \infty), \\ u(x,0) = u_0(x) & \text{in } B_R(0) \times \{t = 0\}. \end{cases}
$$

The starting point is the following result:

Lemma 4.6 *For any $\varepsilon > 0$, $K \in \mathbb{R}$ and $u_0 \in C^1(\overline{B_R(0)})$ radial, there exists a classical solution u_ε of (4.6) which is C^1 in x up to the parabolic boundary and $u_\varepsilon(x,t)$ is radial in x for any $t > 0$.*

The proof of this Lemma follows from classical parabolic theory since the equation is not degenerate (see [27]).

Step 2. Uniform estimates for the u_ε.
We show uniform bounds for the family $(u_\varepsilon)_{\varepsilon > 0}$ and their gradients. To this end, we introduce a smooth supersolution of the form

$$
W(x,t) = C_2|x|^2 + C_1 t + C_0,
$$

where $C_2, C_1 \in \mathbb{R}$ and $C_0 = \sup_{\overline{B_R(0)}} |u_0|$ in order to ensure that $W(\cdot, 0) \geqslant u_\varepsilon(\cdot, 0)$. Since

$$
DW(x,t) = 2C_2 x, \quad D^2 W = 2C_2\, I,
$$

it is sufficient to choose $C_2 > K$ in order to have

$$
\frac{\partial W}{\partial n} = 2C_2 R > \frac{\partial u_\varepsilon}{\partial n} = K, \quad \text{on} \quad \partial B_R(0) \times [0,T],
$$

and $C_1 > 2NC_2$ to ensure that W is a strict supersolution of the problem. Therefore, the maximum of $u_\varepsilon - W$ on $\overline{B_R(0)} \times [0,T]$ is achieved at $t = 0$ and it follows that

$$
u_\varepsilon \leqslant W \leqslant C_0 + C_1 T + C_2 R^2 \quad \text{in} \quad \overline{B_R(0)} \times [0,T].
$$

Since a lower estimate follows by the same way using a subsolution, we get the L^∞ bound (recall that the constants depend only on $\sup_{\overline{B_R(0)}} |u_0|$ and K).

Let us now prove the gradient bound. For any $\varepsilon > 0$, define φ_ε by $u_\varepsilon(x,t) = \varphi_\varepsilon(|x|,t)$ and $\psi_\varepsilon = \partial_r \varphi_\varepsilon$. Then ψ_ε satisfies $P_\varepsilon(\psi_\varepsilon) = 0$ (see (4.1)), and let $C = C(R,K) := \max\{\sup_{\overline{B_R(0)}} |Du_0(x)|,\ |K|\}$. Then by the comparison result obtained in Proposition 4.2, $|\psi_\varepsilon| \leqslant C$ since C is a solution of (4.1), which gives the uniform gradient bound.

Step 3. Extraction of a convergent subsequence.
The previous bounds prove equi-continuity in the x-variable. To show the equi-continuity in the t-variable, we invoke the classical relationship between space continuity and time continuity for solutions of parabolic equations (see [6, Lemma 9.1] for example). Thanks to this argument, we obtain a time modulus of continuity for the u_ε. Hence the family (u_ε) is equi-continuous in $\overline{B_R(0)} \times [0,T]$ and equi-bounded, so that we can extract a subsequence (u_{ε_n}) converging locally uniformly to some $u \in C(\overline{B_R(0)} \times [0,T])$.

Step 4. The limit function u is a classical solution of (4.5).
In order to prove this result, let us notice first that, by the local uniform convergence, standard arguments show that u is a viscosity solution of (4.5) (see for example [14] for a proof). We continue by showing that, using the previous gradient bounds, we can replace the equation in (4.5) by a uniformly parabolic one such that u is still a solution of the new equation.

If $p \in \mathbb{R}^N, |p| \leqslant C$ and $\|\cdot\|$ denotes the Euclidean norm in the space $\mathcal{S}_N$ of symmetric matrices, then we have

$$\|b(p)\| \geqslant \|I\| - \frac{\|p \otimes p\|}{1+|p|^2} \geqslant N - \frac{N|p|^2}{1+|p|^2} \geqslant \frac{N}{1+C} := \lambda > 0.$$

We then define $\phi : \mathcal{S}_N^+ \to \mathcal{S}_N^+$ by $\phi(M) = \psi(\|M\|)M + (1 - \psi(\|M\|))\lambda I$, where $\mathcal{S}_N^+$ is the space of positive symetric matrices and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth nondecreasing function which is 0 at 0 and 1 in $[\lambda, +\infty)$. Replacing the diffusion matrix b in (4.5) by $a = \phi \circ b \in C^\infty(\mathbb{R}^N; \mathcal{S}_N^+)$, we obtain that the new equation is uniformly parabolic. But, since $|Du_\varepsilon| \leqslant C$ by the previous gradient bounds, we have that $a(Du_\varepsilon) = b(Du_\varepsilon)$; thus u is a viscosity solution of the new problem (4.5) with diffusion a. The theory about uniformly parabolic equations (see [27, Theorem 13.25, p.351]) provides

Lemma 4.7 *There exists a unique viscosity solution of the Neumann problem:*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{Tr}[a(Du)D^2u] = 0 & \text{in } B_R(0) \times (0,+\infty), \\ \dfrac{\partial u}{\partial \nu}(x,t) = f & \text{on } \{|x| = R\} \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } B_R(0) \times \{t = 0\}, \end{cases}$$

where a is uniformly elliptic, $u_0 \in C^1(\overline{B_R(0)})$ and $f \in C(\partial B_R(0) \times [0,T])$. Moreover, the solution u is classical (in particular u is C^1 in x up to the boundary).

Now, we since uniqueness holds for viscosity solution of the previous problem (see [3]), u is in fact a classical solution of (4.5). It ends the proof. ■

4.3 Proof of Theorem 4.1

We start with some estimates for the solutions of (1.1) which come from the works of Ecker and Huisken (see [18, Theorems 2.3 and 5.1]).

Proposition 4.8 *There exists a constant* $C = C\big(R, T, \|u_0\|_{C^1(B_{2R}(x_0))}\big)$ *such that for any solution* u *of (1.1),*

$$\sup_{B_R(x_0)\times[0,T]} |u|, \quad \sup_{B_R(x_0)\times[0,T]} |Du| \leqslant C. \tag{4.7}$$

Proof of Theorem 4.1. Let u be any radial solution of (1.1) with initial data u_0. We proceed in several steps:

Step 1. For $R > 1$, let $C = C\big(R, T, \|u_0\|_{C^1(B_{2R}(x_0))}\big)$ be the constant defined in Proposition 4.8 and u_R be the solution built in Proposition 4.4 with $K = -C$. It is clear that u_R is constructed independently of any other solution u, a remark which will be important in the following. By comparison on the derived problem (Proposition 4.2), we obtain

$$\partial_r u \geqslant \partial_r u_R \quad \text{in} \quad \overline{B_R(0)} \times [0, T]. \tag{4.8}$$

Step 2. Now we let R increase to $+\infty$: by the local bounds (4.7), the family $(u_R)_{R\geqslant 1}$ is locally uniformly bounded in C^1, and thus locally equi-continuous. We can then extract a subsequence converging locally uniformly in $\mathbb{R}^N \times [0, \infty)$. Let us call $u^{g,-}$ the limit obtained after extraction.

Step 3. We proceed as in the Step 4 of Proposition 4.4: the limit $u^{g,-}$ turns out to be a viscosity solution of (1.1). Moreover, by the previous gradient bounds, in any fixed ball $B_{R_0}(0)$, for $R \geqslant R_0$, the u_R are solutions of

$$\frac{\partial u}{\partial t} - \mathrm{Tr}[a(Du)D^2 u] = 0 \quad \text{in } B_{R_0}(0) \times (0, +\infty), \tag{4.9}$$

where the diffusion matrix a is defined as in the proof of Proposition 4.4, Step 4, using $C = C\big(R, T, \|u_0\|_{C^1(B_{R_0}(x_0))}\big)$. Thus, $u^{g,-}$ is also a viscosity solution of (4.9) which is classical in $B_{R_0}(0) \times (0, +\infty)$ for any $R_0 \geqslant 1$ by Lemma 4.7; therefore $u^{g,-}$ is classical in $\mathbb{R}^N \times (0, +\infty)$. Finally, as $R \to \infty$ along the chosen subsequence, (4.8) yields in the limit:

$$\partial_r u \geqslant \partial_r u^{g,-} \quad \text{in } \mathbb{R}^N \times [0, +\infty).$$

Note that by construction, $u^{g,-}$ is radial and as was said, is constructed independently of any solution. Hence $u^{g,-}$ is the solution which has minimal gradient among radial solutions.

Step 4. The same argument works for construction of a solution with maximal gradient in the class of radial solutions. It consists in solving the Neumann problem with boundary data $K = +C\big(R, T, \|u_0\|_{C^1(B_{2R}(x_0))}\big)$, and the rest of the proof follows as in the previous steps. ■

5 Proof of the uniqueness result (Theorem 1.1)

Let us finally conclude with our uniqueness result:

Proof of Theorem 1.1. By definition of the maximal and minimal solutions, it is sufficient to prove that $u^+ \equiv u^-$ to obtain uniqueness in the class of continuous viscosity solutions. Moreover, from Proposition 2.3, we can suppose that the radial function u_0 is in $C^1(I\!R^N)$ without loss of generality. Therefore, we know from Section 4 that there exist radial extremal gradient solutions $u^{g,+}$ and $u^{g,-}$.

Step 1. We first claim that $\varphi^{g,+} = \varphi^-$. Indeed, suppose there exists $(\bar{r}, \bar{t}) \in [0, +\infty) \times [0, +\infty)$ such that

$$\varphi^{g,+}(\bar{r}, \bar{t}) - \varphi^-(\bar{r}, \bar{t}) > \varepsilon, \tag{5.1}$$

for some $\varepsilon > 0$. By Theorem 4.1, $\partial_r(\varphi^{g,+} - \varphi^-)(r, \bar{t}) \geqslant 0$ for any $r \geqslant \bar{r}$; therefore, there exists $r_0 \geqslant 1$ such that (5.1) holds for $(r, \bar{t}) \in [r_0, +\infty) \times \{\bar{t}\}$. Applying Lemma 3.1 with r_0 and $T = \bar{t}$, we obtain

$$\int_{r_0}^{+\infty} (\varphi^+ - \varphi^-)(r, \bar{t}) dr \geqslant \int_{r_0}^{+\infty} (\varphi^{g,+} - \varphi^-)(r, \bar{t}) dr \geqslant \int_{r_0}^{+\infty} \varepsilon dr,$$

which is absurd. Then, similar arguments also prove that $\varphi^{g,-} = \varphi^+$.

Step 2. Now let us fix $T > 0$ and set for any $c > 0$

$$U_{c,T}(x, t) := u^+(x, t) - u^-(x, t) - \frac{c}{T - t} \quad \text{for } (x, t) \in I\!R^N \times [0, T).$$

It follows from Step 1 that $U_{c,T}(x, t) = \varphi^{g,-}(|x|, t) - \varphi^{g,+}(|x|, t) - c/(T - t)$ is nonincreasing with respect to $|x|$ and thus that $M = \sup_{I\!R^N \times [0,T)} U_{c,T}$ is achieved at some point $(0, t_0)$ with $t_0 \in [0, T)$.

If $t_0 > 0$ we have a interior maximum point and thus

$$\partial_t u^+(0, t_0) - \partial_t u^-(0, t_0) = \frac{c}{(T - t)^2}, \quad Du^+(0, t_0) = Du^-(0, t_0),$$

and $D^2 u^+(0, t_0) \leqslant D^2 u^-(0, t_0)$. Thus, using the equation under the form $\partial_t u - \mathrm{Tr}(b(Du) D^2 u) = 0$, we obtain

$$\frac{c}{(T - t)^2} = \mathrm{Tr}\left[b(Du^+)(D^2 u^+ - D^2 u^-)\right] \leqslant 0,$$

hence we reach a contradiction.

It follows that $t_0 = 0$ and thus $M < 0$; hence

$$u^+(x, t) - u^-(x, t) \leqslant \frac{c}{T - t} \quad \text{for } (x, t) \in I\!R^N \times [0, T).$$

Letting c go to 0^+, it gives the result since T is arbitrary. ■

6 A remark on the stationary problem

Our aim in this section is to compare the uniqueness result for (1.1) with those we can obtain for the associated stationary problem, namely for the elliptic equation

$$-\Delta u + \frac{\langle D^2uDu, Du\rangle}{1+|Du|^2} + \lambda u = f \quad \text{in} \quad \mathbb{R}^N. \tag{6.1}$$

This equation arises naturally when considering the semi-group approach to the parabolic equation [15], and we especially consider its radial version

$$-\frac{\partial_{rr}\varphi}{1+(\partial_r\varphi)^2} - (N-1)\frac{\partial_r\varphi}{r} + \lambda\varphi = f(r) \quad \text{in } (0,+\infty), \tag{6.2}$$

where $\lambda > 0$ and $f \in C(\mathbb{R}^N)$ is a radial function we identify with its restriction to any half-line.

In fact, we will prove that the uniqueness result for C^2 radial solutions of (6.2) takes place under growth conditions and more precisely in the following class of functions

$$\mathcal{C} = \left\{\varphi \in C^2([0,+\infty)) : \varphi(r)\mathrm{e}^{-\lambda r^2/2(N-1)} \underset{r\to+\infty}{\longrightarrow} 0\right\}.$$

The result is the following:

Theorem 6.1 *For $N > 1$ and a radial function f, if $\varphi_1 \in \mathcal{C}$ (respectively $\varphi_2 \in \mathcal{C}$) is a subsolution (respectively a supersolution) of (6.2) then*

$$\varphi_1 \leqslant \varphi_2 \quad \text{in } [0,+\infty).$$

In particular, (6.1) has at most one radial solution in $\mathcal{C}$.

Moreover, this result is optimal in the sense that we have a counter-example to uniqueness when the solutions are allowed to have the critical growth: consider $\varphi(r) = \mathrm{e}^{\lambda r^2/2(N-1)}$. A straightforward computation shows that φ is a classical solution of (6.2) on $[0,+\infty)$ with

$$f(r) = -\frac{\frac{\lambda}{N-1}\left(\frac{\lambda r^2}{N-1}+1\right)\mathrm{e}^{\lambda r^2/2(N-1)}}{1+\left(\frac{\lambda r}{N-1}\right)^2 \mathrm{e}^{2\lambda r^2/2(N-1)}}.$$

It is clear that $f \in BUC([0,+\infty)) \cap C^\infty([0,+\infty))$ can be extended to $\mathbb{R}^N$ as a smooth radial function. However, we know from the theory of viscosity solutions, that for such a data f there is a unique radial viscosity solution $\tilde{\varphi}$ of (6.1) in $BUC(\mathbb{R}^N)$, and from the usual machinery for elliptic equations (see *e.g* [21]), this solution is a classical one. Since $\tilde{\varphi} \neq \varphi$, this proves that nonuniqueness may occur outside the class $\mathcal{C}$.

Before giving the proof of the theorem, we state and prove a lemma which will be useful.

Lemma 6.2 *Consider an elliptic pde under the form*

$$F(Du, D^2u) + \lambda u = f \quad in \quad \mathbb{R}^N, \tag{6.3}$$

where $\lambda > 0$, F *is a continuous elliptic nonlinearity and* $f \in C(\mathbb{R}^N)$. *Let* u_1 *(respectively* u_2*) be a* C^2 *subsolution (respectively supersolution) of (6.3). Suppose that there exists* $x_0 \in \mathbb{R}^N$ *such that* $u_1(x_0) > u_2(x_0)$. *Then for all* $r > |x_0|$,

$$\max_{\overline{B_r(0)}}\{u_1 - u_2\} = \max_{\partial B_r(0)}\{u_1 - u_2\}.$$

In particular, if $u_1(x) = \varphi_1(|x|)$ *(respectively* $u_2(x) = \varphi_2(|x|)$*) are radial then* $(\varphi_1 - \varphi_2)(r) \geqslant (\varphi_1 - \varphi_2)(s) > 0$ *for all* $r \geqslant s \geqslant |x_0|$.

Proof of Lemma 6.2. Let $r \geqslant |x_0|$ and consider

$$\max_{\overline{B_r(0)}}\{u_1 - u_2\}, \tag{6.4}$$

which is positive by assumption. Suppose that this maximum is achieved at x_1 such that $0 \leqslant |x_1| < r$. In this case,

$$D(u_1 - u_2)(x_1) = 0 \quad \text{and} \quad D^2(u_1 - u_2)(x_1) \leqslant 0. \tag{6.5}$$

Writing the Equation (6.3) for u_1 and u_2 at x_1, and using (6.5) we get

$$\lambda(u_1(x_1) - u_2(x_1)) \leqslant F(Du_2(x_1), D^2u_2(x_1)) - F(Du_1(x_1), D^2u_1(x_1)) \leqslant 0$$

by ellipticity. Thus

$$0 \geqslant u_1(x_1) - u_2(x_1) \geqslant u_1(x_0) - u_2(x_0) > 0,$$

which is a contradiction. This proves that the maximum in (6.4) is achieved on $\partial B_r(0)$. In particular in the radial case, for every $s \in [0, r]$,

$$0 < (\varphi_1 - \varphi_2)(s) \leqslant (\varphi_1 - \varphi_2)(r),$$

which ends the proof of the lemma. ∎

We turn to the proof of the theorem.

Proof of Theorem 6.1. We argue by contradiction, assuming that there exists $r_0 \geqslant 0$ such that

$$(\varphi_1 - \varphi_2)(r_0) = \varepsilon > 0. \tag{6.6}$$

Note that, from Lemma 6.2, we can suppose that r_0 is as large as we want. We take $r_0 > 0$ such that

$$\lambda r_0 - \frac{N-1}{r_0} > 0. \tag{6.7}$$

Writing Equation (6.2) for the subsolution φ_1 and the supersolution φ_2 at any point $r \geqslant r_0$, substracting the inequalities and integrating on $[r_0, r]$, we get

$$-(N-1)\int_{r_0}^{r} \frac{\partial_r(\varphi_1-\varphi_2)(\sigma)}{\sigma}d\sigma + \lambda\int_{r_0}^{r}(\varphi_1-\varphi_2)(\sigma)d\sigma \leqslant 2\pi$$

Defining $\psi(r) = (\varphi_1 - \varphi_2)(r)/r$ and using a integration by parts, we obtain, for every $r \geqslant r_0$,

$$-(N-1)\psi(r) + (N-1)\psi(r_0) + \int_{r_0}^{r}\psi(\sigma)\Big(\lambda\sigma - \frac{N-1}{\sigma}\Big)\,d\sigma \leqslant 2\pi. \tag{6.8}$$

For $r \geqslant r_0$, we define

$$H(r) = (N-1)\psi(r_0) - 2\pi + \int_{r_0}^{r}\psi(\sigma)\Big(\lambda\sigma - \frac{N-1}{\sigma}\Big)\,d\sigma.$$

Let us suppose first that

$$(N-1)\psi(r_0) - 2\pi > 0. \tag{6.9}$$

From Lemma 6.2 and (6.7), we have $\psi(\sigma)(\lambda\sigma - (N-1)/\sigma) > 0$ for every $\sigma \geqslant r_0$. Then it follows that $H(r) > 0$ for every $r \geqslant r_0$. From (6.8), we obtain, for every $r \geqslant r_0$,

$$\frac{1}{N-1}H(r) \leqslant \psi(r) \tag{6.10}$$

and multiplying by $\lambda r - (N-1)/r > 0$, we get

$$\frac{1}{N-1}H(r)\Big(\lambda r - \frac{N-1}{r}\Big) \leqslant \psi(r)\Big(\lambda r - \frac{N-1}{r}\Big) = H'(r). \tag{6.11}$$

Integrating the ordinary differential inequality given by (6.11), we obtain

$$H(r) \geqslant \frac{K}{r}\mathrm{e}^{\lambda r^2/2(N-1)}$$

for some positive constant $K = K(r_0, N)$. From (6.10), we get

$$(\varphi_1-\varphi_2)(r) \geqslant \frac{K}{N-1}\mathrm{e}^{\lambda r^2/2(N-1)}, \quad \forall r \geqslant r_0,$$

which leads to a contradiction by letting r go to $+\infty$ since $\varphi_1, \varphi_2 \in \mathcal{C}$.

Finally (6.9) is absurd and thus, for every $r \geqslant r_0$, $\psi(r) \leqslant 2\pi/(N-1)$. Now, from (6.8), it follows

$$\begin{aligned}\int_{r_0}^{r}\sigma\psi(\sigma)\,d\sigma &= \int_{r_0}^{r}(\varphi_1-\varphi_2)(\sigma)\,d\sigma\\ &\leqslant 2\pi + (N-1)\psi(r) - (N-1)\psi(r_0) + \int_{r_0}^{r}\psi(\sigma)\frac{N-1}{\sigma}\,d\sigma\\ &\leqslant 4\pi + \int_{r_0}^{r}\frac{4C}{\sigma}\,d\sigma = 4\pi + 2\pi\ln\Big(\frac{r}{r_0}\Big).\end{aligned}$$

But, using (6.6) we obtain, for every $r \geqslant r_0$,

$$\varepsilon(r-r_0) \leqslant \int_{r_0}^{r} (\varphi_1 - \varphi_2)(\sigma)\,d\sigma \leqslant 4\pi + 2\pi \ln\left(\frac{r}{r_0}\right),$$

which leads again to a contradiction when r goes to $+\infty$. It completes the proof of the theorem. ■

7 Application to the mean curvature motion

We state here a consequence of Theorem 1.1 for the evolution by mean curvature of an entire graph.

Theorem 7.1 *Let $\Gamma_0 = \mathrm{Graph}(u_0)$ where $u_0 \in C(\mathbb{R}^N)$ is a radial function. Then the generalized evolution by mean curvature $(\Gamma_t, \Omega_t^+)_{t\geqslant 0}$ of (Γ_0, Ω_0^+) is given by*

(7.1) $\quad \Gamma_t = \mathrm{Graph}(u(\cdot,t))$ *and* $\Omega_t^+ = \{y > u(x,t)\}$ *for any* $t \geqslant 0$,

where $u \in C^\infty(\mathbb{R}^N \times (0,+\infty)) \cap C(\mathbb{R}^N \times [0,+\infty))$ is the unique continuous viscosity solution of (1.1). Therefore $(\Gamma_t)_{t\geqslant 0}$ does not develop any interior and, for $t > 0$, the generalized evolution coincides with the classical mean curvature flow in the sense of differential geometry.

Proof of Theorem 7.1. The statement (7.1) is obvious since, by Theorem 1.1, the functions u^+ and u^- defined in Theorem 2.1, coincide: $u^+ = u^- = u \in C^\infty(\mathbb{R}^N \times (0,+\infty)) \cap C(\mathbb{R}^N \times [0,+\infty))$.

We have shown that Γ_t is a smooth hypersurface. To prove that Γ_t evolves by mean curvature, we have to prove that the normal velocity $\mathcal{V}^{\Gamma_t}_{(x_0,t_0)}$ at each $(x_0, u(x_0,t_0)) \in \Gamma_{t_0}, t_0 > 0$, is equal to $-\mathrm{div}_{(x,y)}(n)(x_0,t_0)$, where $n(x_0,t_0)$ is the outward unit normal to $\mathrm{Epi}(u(\cdot,t_0)) = \{(x,y) \in \mathbb{R}^{N+1} : y \geqslant u(x,t_0)\}$ at the point $(x_0, u(x_0,t_0))$. It is a classical calculation we recall for reader's convenience. Let $P(t) = (x(t), u(x(t),t))$ be a smooth curve on the front such that $x(t_0) = x_0$. Noticing that $n(x_0,t_0) = (Du(x_0,t_0),-1)/\sqrt{1+|Du(x_0,t_0)|^2}$, we obtain that the normal velocity of $P(t)$ at $t = t_0$ is:

$$\begin{aligned}
\mathcal{V}^{\Gamma_t}_{(x_0,t_0)} &= \left\langle \frac{dP(t_0)}{dt}, n(x_0,t_0) \right\rangle = \frac{-\frac{\partial u}{\partial t}(x_0,t_0)}{\sqrt{1+|Du(x_0,t_0)|^2}} \\
&= \frac{-1}{\sqrt{1+|Du(x_0,t_0)|^2}} \mathrm{Tr}\left[\left(I - \frac{Du(x_0,t_0)\otimes Du(x_0,t_0)}{1+|Du(x_0,t_0)|^2}\right) D^2u(x_0,t_0)\right] \\
&= -\mathrm{div}_x\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)(x_0,t_0) \\
&= -\mathrm{div}_{(x,y)}\left(\frac{(Du,-1)}{\sqrt{1+|Du|^2}}\right)(x_0,u(x_0,t_0),t_0) \\
&= -\mathrm{div}_{(x,y)}(n)(x_0,t_0),
\end{aligned}$$

which proves the claim. □

We end with some comments about the "fattening phenomena". This issue is of interest since it is closely related to the nonuniqueness of the generalized evolution by mean curvature of hypersurfaces when starting with an hypersurface $\Gamma_0 \subset I\!R^N$. There are some situations where Γ_0 develops an interior: see Evans and Spruck [19], Soner [29], Barles, Soner and Souganidis [7], Ilmanen [23], Barles and Souganidis [8] and the discussion in [6]. However, the question: may a graph $\Gamma_0 = \mathrm{Graph}(u_0)$ develops an interior? is an open question in the whole generality. As we said in the introduction and in Section 2, it is equivalent to prove uniqueness for (1.1).

We have two types of results. For the first type, we prove first the uniqueness for (1.1) and then derive the non fattening for the generalized evolution in the following situations: (i) here, when Γ_0 is the graph of a radial function; (ii) in [5] in dimension 1, i.e. when $u_0 \in C(I\!R)$; (iii) in [4], when $u_0 \in C(I\!R^N), N \geqslant 1$, with some polynomial-type restrictions on the gradient of u_0.

The second type of results consists in proving that Γ_t does not develop any interior to get uniqueness results for (1.1). We obtain this result in the following cases: (i) when $\Gamma_0 = \mathrm{Graph}(u_0)$ is the graph of a convex function in [6]; (ii) when u_0 is "convex at infinity," see [9].

References

[1] D. G. Aronson. Widder's inversion theorem and the initial distribution problem. *SIAM J. Math. Anal.*, 12:639–651, 1981.

[2] D. G. Aronson, M. G. Crandall, and L. A. Peletier. Stabilization of solutions of a degenerate nonlinear diffusion problem. *Nonlinear Anal.*, 6(10):1001–1022, 1982.

[3] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi.* Springer-Verlag, Paris, 1994.

[4] G. Barles, S. Biton, M. Bourgoing, and O. Ley. Uniqueness results for quasi-linear parabolic equations through viscosity solutions' methods. *To appear in Calc. Var.*

[5] G. Barles, S. Biton, and O. Ley. Quasilinear parabolic equations, unbounded solutions and geometrical equations. Uniqueness without growth conditions and applications to the mean curvature flow in $I\!R^2$. *To appear in J. Differential Equations.*

[6] G. Barles, S. Biton, and O. Ley. A geometrical approach to the study of unbounded solutions of quasilinear parabolic equations. *Arch. Rational Mech. Anal.*, 162:287–325, 2002.

[7] G. Barles, H. M. Soner, and P. E. Souganidis. Front propagation and phase field theory. *SIAM J. Control Optim.*, 31(2):439–469, 1993.

[8] G. Barles and P. E. Souganidis. A new approach to front propagation problems: theory and applications. *Arch. Rational Mech. Anal.*, 141(3):237–296, 1998.

[9] S. Biton, P. Cardaliaguet, and O. Ley. Non fattening conditions for the generalized evolution by mean curvature. *In preparation*, 2002.

[10] H. Brezis. Semilinear equations in $\mathbb{R}^n$ without condition at infinity. *Appl. Math. Optim.*, 12:271–282, 1984.

[11] E. Chasseigne and J.-L. Vazquez. Theory of extended solutions for fast diffusion equations in optimal classes of data. Radiation from singularities. *To appear in Arch. Rational Mech. Anal.*, 2002.

[12] Y. G. Chen, Y. Giga, and S. Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.

[13] K.-S. Chou and Y.-C. Kwong. On quasilinear parabolic equations which admit global solutions for initial data with unrestricted growth. *Calc. Var.*, 12:281–315, 2001.

[14] M. G. Crandall, H. Ishii and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[15] M. G. Crandall and T. Liggett. Generation of semi-groups of nonlinear transformations on general Banach spaces. *Amer. J. Math.*, 93:265–293, 1971.

[16] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 277(1):1–42, 1983.

[17] B. E. J. Dahlberg and C. E. Kenig. Non-negative solutions of generalized porous medium equations. *Rev. Mat. Ibero.*, 2(3):267–305, 1986.

[18] K. Ecker and G. Huisken. Interior estimates for hypersurfaces moving by mean curvature. *Invent. Math.*, 105(3):547–569, 1991.

[19] L. C. Evans and J. Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3):635–681, 1991.

[20] A. Friedman. *Partial differential equations of parabolic type.* Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[21] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Springer-Verlag, Berlin, second edition, 1983.

[22] M. A. Herrero and M. Pierre. The Cauchy problem for $u_t = \delta u^m$ when $0 < m < 1$. *Trans. Amer. Math. Soc.*, 291(1):145–158, 1985.

[23] T. Ilmanen. Elliptic regularization and partial regularity for motion by mean curvature. *Mem. Amer. Math. Soc.*, 108(520), 1994.

[24] H. Ishii. Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations. *Indiana Univ. Math. J.*, 33(5):721–748, 1984.

[25] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type.* American Mathematical Society, Providence, R.I., 1967. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23.

[26] O. Ley. Lower-bound gradient estimates for first-order Hamilton-Jacobi equations and applications to the regularity of propagating fronts. *Adv. Differential Equations*, 6(5):547–576, 2001.

[27] G. M. Lieberman. *Second order parabolic differential equations.* World Scientific Publishing Co. Pte. Ltd., Singapore, 1996.

[28] S. Osher and J. Sethian. Fronts propagating with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations. *J. Comp. Physics*, 79:12–49, 1988.

[29] H. M. Soner. Motion of a set by the curvature of its boundary. *J. Differential Equations*, 101:313–372, 1993.