Uniqueness for parabolic equations without growth condition and applications to the mean curvature flow in $I\!R^2$

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Abstract

In this article, we prove a comparison result for viscosity solutions of a certain class of fully nonlinear, possibly degenerate, parabolic equations; the main new feature of this result is that it holds for any, possibly discontinuous, solutions without imposing any restrictions on their growth at infinity. The main application of this result which was also our main motivation to prove it, is the uniqueness of solutions to one-dimensional equations including the mean curvature equation for graphs without assuming any restriction on their behavior at infinity.

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1 Introduction

The main motivation of this paper comes from the following, rather surprising, result of Ecker and Huisken [13]: for any initial data $u_0 \in W^{1,\infty}_{\rm loc}(I\!R^N)$, there exists a smooth solution of the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \dfrac{\langle D^2uDu, Du\rangle}{1+|Du|^2} = 0 & \text{in } I\!R^N \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{in } I\!R^N\,. \end{cases} \tag{1}$$

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Here and below the solution u is a real-valued function, Du and D^2u denote respectively its gradient and Hessian matrix while $|\cdot|$ and $\langle \cdot, \cdot \rangle$ stand for the classical Euclidean norm and inner product in $I\!R^N$.

In order to be complete, it is also worth mentioning that, following Angenent [1], one can extend this result to merely continuous initial data using the interior gradient estimates of Evans and Spruck [15] (see [6]).

The very non-standard feature of this kind of results is that no assumption on the behavior of the initial data at infinity is imposed and therefore the solutions may also have any possible behavior at infinity. A natural and intriguing question is then whether such a solution is unique or not.

This question leads us to study of the uniqueness properties for unbounded viscosity solutions of quasilinear, possibly degenerate, parabolic equations set in $I\!R^N$ and this article is a continuation of this program started in [6] (see also [5], [4] and [7]).

In this paper, we are more particularly interested in the one-dimensional case for (1) where this equation reduces to

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{u_{xx}}{1+u_x^2} = 0 & \text{in } I\!R \times (0,T), \\ u(\cdot, 0) = u_0 & \text{in } I\!R. \end{cases} \tag{2}$$

One of the main contribution of this work is a uniqueness result for smooth solutions of (2) (and even for more general equations), without any restriction on their growth at infinity (see Theorem 4.2); it immediately yields a comparison result for, possibly discontinuous, viscosity solutions of (2) by the geometrical approach of [6] (cf. Corollary 5.1), i.e. a complete answer to the question we address. After this work was completed, we learn that Chou and Kwong [9] proved general uniqueness results for a certain class of quasilinear parabolic equations in the one-dimensional case: their hypothesis are of different nature from ours and their methods are quite different but their result, which is also valid for solutions without any restriction on their behavior at infinity, includes also the mean curvature equation.

This kind of uniqueness results is non-standard: it is well-known that uniqueness fails, in general, if we do not impose growth conditions on the solutions at infinity, like, for example, the heat equation. The only (general) case where similar results hold is the one of first-order equations where they are a consequence of "finite speed of propagation" type properties (see Crandall and Lions [12], Ishii [19] and Ley [20]) but they are very unusual for second-order equation.

Our uniqueness proof for (2) consists in integrating the equation, which leads to consider the pde

$$\frac{\partial v}{\partial t} - \arctan(v_{xx}) = 0 \quad \text{in } I\!R \times (0, +\infty). \tag{3}$$

And this is this new equation which is shown to satisfy a comparison principle for viscosity solutions without growth condition at infinity.

We refer the reader to Crandall, Ishii and Lions [11], Fleming and Soner [17], Bardi and Capuzzo Dolcetta [2] or Barles [3], etc. for a presentation of the notion of viscosity solutions and for the key basic results.

In fact, the comparison result we are going to prove and apply to (3) holds in $I\!R^N$ (and not only in dimension 1) and for more general equations of the type

$$\begin{cases} \dfrac{\partial u}{\partial t} + F(x,t,Du,D^2u) = 0 & \text{in } I\!R^N \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } I\!R^N, \end{cases} \tag{4}$$

where F is a continuous function from $I\!R^N \times [0,T] \times I\!R^N \times \mathcal{S}_N$ into $I\!R$ and satisfies suitable assumptions (see **(H1)** and **(H2)** in Section 2). Our proof relies essentially on the use of a "friendly giant method." This result implies the uniqueness for smooth solutions of quite general equations in the one-dimension space including (2) as we mention it above (see Section 4).

Finally we investigate the consequences of the existence and uniqueness of solutions for (2) from the geometrical point of view. The geometrical interpretation of (2) is that the graph of u at every time $t \geq 0$, namely

$$\Gamma_t := \mathrm{Graph}(u(\cdot,t)),$$

is a smooth hypersurface of $I\!R^2$ which evolves by its mean curvature. Such evolution has been studied recently by the so-called "level-set approach": introduced for numerical purposes by Osher and Sethian [21], this approach was developed by Evans and Spruck [14] and Chen, Giga and Goto [8]; the generalized motion of hypersurfaces is described through the evolution of the level-sets of solutions of a suitable geometrical pde. This approach offers a lot of advantages: it is possible to deal with nonsmooth hypersurfaces, it allows numerical computations, etc. The main issue is the agreement with the classical motion defined in differential geometry. These questions were addressed by a lot of authors (see especially Evans and Spruck [14], [15] and Ilmanen [18]).

Our uniqueness result allows us to show that, in the case of the mean curvature motion of any entire continuous graph in $I\!R^2$, the level-set approach agrees with the classical motion defined in differential geometry. As a by-product, we have a comparison result between possibly discontinuous viscosity sub and supersolutions of (2) and therefore the uniqueness holds even in the class of discontinuous solutions. We refer to Section 5 for the complete statement of these results.

The paper is divided as follows. In Section 2, we prove the strong comparison principle for viscosity solutions of equation (4). Section (3) is devoted to the proof of the existence of a solution for (4). In Section 4, we apply the previous result to uniqueness for some one-dimensional equations including (2) as a particular case. In the last section, we state some geometrical consequences for the mean curvature flow in $I\!R^2$ and more general geometrical motions.

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2 A comparison result for viscosity solutions without growth conditions at infinity

In order to state the comparison principle for viscosity solutions of equation (4), we use the following assumptions in which $\bar{B}(x_0, R) = \{x \in I\!R^N : |x - x_0| \leq R\}$ denotes the ball of radius $R > 0$ centered in $x_0 \in I\!R^N$ and $\mathcal{S}_N$ the space of $N \times N$ symmetric matrices; moreover, for every symmetric matrix $X = P(\text{diag}(\lambda_i)_{1 \leq i \leq N})P^T$, where the λ_i's are the eigenvalues of X and P is orthogonal, we define $X^+ = P(\text{diag}(\lambda_i^+)_{1 \leq i \leq N})P^T$.

(H1) For any $R > 0$, there exists a function $m_R : I\!R_+ \to I\!R_+$ such that $m_R(0^+) = 0$ and

$$F(y, t, \eta(x-y), Y) - F(x, t, \eta(x-y), X) \leq m_R\left(\eta|x-y|^2 + |x-y|\right),$$

for all $x, y \in \bar{B}(0, R)$, $t \in [0, T]$, $X, Y \in \mathcal{S}_N$, and $\eta > 0$ such that

$$-3\eta \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\eta \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

(H2) There exists $0 < \alpha < 1$ and constants $K_1 > 0$ and $K_2 > 0$ such that

$$F(x, t, p, X) - F(x, t, q, Y) \leq K_1|p - q|(1 + |x|) + K_2(\text{Tr}[Y - X]^+)^\alpha,$$

for every $(x, t, p, X, Y) \in I\!R^N \times I\!R^N \times [0, T] \times \mathcal{S}_N \times \mathcal{S}_N$.

Our result is the following

Theorem 2.1 *Assume that **(H1)** and **(H2)** hold and let u (respectively v) be an upper-semicontinuous viscosity subsolution (respectively lower-semicontinuous viscosity supersolution) of (4). If $u(\cdot, 0) \leq v(\cdot, 0)$ in $I\!R^N$, then $u \leq v$ in $I\!R^N \times [0, +\infty)$.*

Before proving this result, let us comment the assumptions. Assumption **(H1)** is the classical one to ensure uniqueness for this type of fully nonlinear parabolic equations (see Condition (3.14) of [11]): here we just state it in a more local way. The assumption **(H2)** contains all the restriction which allows such a result to hold: the first term of the right-hand side is the classical one to obtain "finite speed of propagation" type results while the second one concerns the behavior of F in D^2u, and in particular at infinity as (3) shows it. Obviously such an assumption or a related one is needed since such a result cannot be true for the heat equation. A nonlinearity F which satisfies **(H2)** is necessarily of the form $F_1(x,t,p)+F_2(x,t,X)$. We also point out that **(H2)** implies that F is degenerate elliptic. We do not know if these assumptions are close or far to be optimal.

As an example of pde which satisfies **(H1)**–**(H2)**, we have

$$\frac{\partial u}{\partial t} - \left[(a(x,t)\Delta u)^+\right]^\alpha + b(x,t)|Du| = f(x,t) \quad \text{in } I\!\!R^N \times (0,T),$$

where a,b are continuous functions on $I\!\!R^N \times (0,T)$, Lipschitz continuous in x uniformly with respect to t, $f \in C(I\!\!R^N \times (0,T))$ and $0 < \alpha < 1$.

Now we turn to the proof of the theorem.

Proof of Theorem 2.1. The proof is done in two steps: the first one is a kind of linearization procedure which yields a new pde for $u-v$. The second one consists in the construction of a suitable smooth supersolution of this new pde which blows up at the boundary of balls.

The first step is described in the

Lemma 2.1 *Under the assumptions of Theorem 2.1, the upper-semicontinuous function $\omega := u - v$ is a viscosity subsolution of*

$$A[\omega] = \frac{\partial \omega}{\partial t} - K_1(1+|x|)|D\omega| - K_2\left[Tr(D^2\omega)^+\right]^\alpha = 0 \quad \textit{in } I\!\!R^N \times (0,+\infty). \tag{5}$$

Moreover $\omega(\cdot,0) \le 0$ in $I\!\!R^N$.

Then the second step relies on the

Lemma 2.2 *There exists $c > 0$ and $k > \beta > 0$ such that, for $R > 0$ large enough, the smooth function*

$$\chi_R(x,t) = \varphi\left((1+R^2)^{\frac{1}{2}}(1-ct) - (1+|x|^2)^{\frac{1}{2}}\right) \quad \textit{with } \varphi(r) = R^\beta/r^k, \tag{6}$$

is a strict supersolution of (5) in the domain

$$\mathcal{D}(c,R) = \left\{(x,t) \in I\!\!R^N \times [0,T]:\ ct < \frac{1}{2},\ (1+|x|^2)^{\frac{1}{2}} < (1+R^2)^{\frac{1}{2}}(1-ct)\right\}. \tag{7}$$

We postpone the proof of the lemmas and we first conclude the proof of the theorem.

We consider $\sup\limits_{\mathcal{D}(c,R)} \{\omega - \chi_R\}$; since χ_R goes to $+\infty$ on the lateral side of $\mathcal{D}(c,R)$, this supremum is achieved at some point $(\bar{x},\bar{t}) \in \mathcal{D}(c,R)$.

We cannot have $\bar{t} > 0$ because, otherwise since by Lemma 2.1 ω is a viscosity subsolution of (5), we would have $A[\chi_R(\bar{x},\bar{t})] \le 0$, which would contradict the fact that χ_R is a strict supersolution of this pde by Lemma 2.2.

Thus, necessarily $\bar{t} = 0$ and the maximum is nonpositive since $\omega(\cdot, 0) \le 0$ and $\chi_R(\cdot, 0) > 0$ in $I\!R^N$. It follows that, for every $(x,t) \in \mathcal{D}(c,R)$,

$$(u-v)(x,t) \le \frac{R^\beta}{[(1+R^2)^{\frac{1}{2}}(1-ct) - (1+|x|^2)^{\frac{1}{2}}]^k} \,. \tag{8}$$

In order to conclude, we let R go to $+\infty$ in this inequality for $t < 1/2c$: since $k > \beta > 0$, we obtain $(u-v)(x,t) \le 0$ for any $x \in I\!R^N$. To prove the same property for all $t \in [0,T]$, we iterate in time and the proof of the theorem is complete. $\square$

We turn to the proof of the lemmas.

Proof of Lemma 2.1. Let $\phi \in C^2(I\!R^N \times (0,+\infty))$ and suppose that $(\bar{x}, \bar{t}) \in I\!R^N \times (0,+\infty)$ is a strict local maximum point of $\omega - \phi = u - v - \phi$ and more precisely a strict maximum point of this function in $\bar{B}(\bar{x},\rho)^2 \times [\bar{t}-\rho, \bar{t}+\rho]$.

We consider, for $\varepsilon > 0$,

$$\max_{\bar{B}(\bar{x},\rho)^2 \times [\bar{t}-\rho,\bar{t}+\rho]} \left\{ u(x,t) - v(y,t) - \phi(x,t) - \frac{|x-y|^2}{\varepsilon^2} \right\} \,. \tag{9}$$

By classical arguments, it is easy to prove that the maximum in (9) is achieved at points $(x_\varepsilon, y_\varepsilon, t_\varepsilon)$ such that

$$(x_\varepsilon, y_\varepsilon, t_\varepsilon) \to (\bar{x}, \bar{x}, \bar{t}) \quad \text{and} \quad \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} \to 0 \quad \text{as } \varepsilon \to 0. \tag{10}$$

Hence, for ε small enough, $(x_\varepsilon, y_\varepsilon, t_\varepsilon) \in B(\bar{x},\rho) \times B(\bar{x},\rho) \times (\bar{t}-\rho, \bar{t}+\rho)$ and following [11], there exist $a, b \in I\!R$, $X, Y \in \mathcal{S}_N$, such that, if we set $p_\varepsilon := \dfrac{2(x_\varepsilon - y_\varepsilon)}{\varepsilon^2}$, we have

$$(a, D\phi(x_\varepsilon, t_\varepsilon) + p_\varepsilon, X + D^2\phi(x_\varepsilon, t_\varepsilon)) \in \bar{\mathcal{P}}^{2,+}u(x_\varepsilon, t_\varepsilon) \,,$$

$$(b, p_\varepsilon, Y) \in \bar{\mathcal{P}}^{2,-}v(x_\varepsilon, t_\varepsilon)$$

and such that $a - b = \dfrac{\partial \phi}{\partial t}(x_\varepsilon, t_\varepsilon)$ and

$$-\frac{6}{\varepsilon^2}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \frac{6}{\varepsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Using that u and v are respectively viscosity sub- and supersolution of (4), we have

$$\frac{\partial \phi}{\partial t}(x_\varepsilon, t_\varepsilon) + F\left(x_\varepsilon, D\phi(x_\varepsilon, t_\varepsilon) + p_\varepsilon, X + D^2\phi(x_\varepsilon, t_\varepsilon)\right) - F\left(y_\varepsilon, p_\varepsilon, Y\right) \le 0.$$

From **(H1)** applied with $R = |\bar{x}| + \rho$, we have

$$-F\left(y_\varepsilon, p_\varepsilon, Y\right) \geq -F\left(x_\varepsilon, p_\varepsilon, X\right) - m_R\left(2\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + |x_\varepsilon - y_\varepsilon|\right)$$

and from **(H2)**, we get

$$\begin{aligned} &F\left(x_\varepsilon, D\phi(x_\varepsilon, t_\varepsilon) + p_\varepsilon, X + D^2\phi(x_\varepsilon, t_\varepsilon)\right) - F\left(x_\varepsilon, p_\varepsilon, X\right) \\ &\qquad \geq -K_1|D\phi(x_\varepsilon, t_\varepsilon)|(1 + |x_\varepsilon|) - K_2\left[\mathrm{Tr}(D^2\phi(x_\varepsilon, t_\varepsilon))^+\right]^\alpha . \end{aligned}$$

Finally, we obtain

$$\begin{aligned} &\frac{\partial\phi}{\partial t}(x_\varepsilon, t_\varepsilon) - K_1|D\phi(x_\varepsilon, t_\varepsilon)|(1 + |x_\varepsilon|) - K_2\left[\mathrm{Tr}(D^2\phi(x_\varepsilon, t_\varepsilon))^+\right]^\alpha \\ &\qquad\qquad \leq m_R\left(2\frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon^2} + |x_\varepsilon - y_\varepsilon|\right) . \end{aligned}$$

Letting ε go to 0 and using (10) yields

$$\frac{\partial\phi}{\partial t}(\bar{x}, \bar{t}) - K_1|D\phi(\bar{x}, \bar{t})|(1 + |\bar{x}|) - K_2\left(\mathrm{Tr}[D^2\phi(\bar{x}, \bar{t})]^+\right)^\alpha \leq 0$$

which is exactly the inequality showing that ω is a subsolution of (5). $\square$

Proof of Lemma 2.2. In order to check that χ is a strict supersolution for a suitable choice of constants $k > \beta > 0$ and $c > 0$ independent of R, we set $r = (1+R^2)^{1/2}(1-ct) - (1 + |x|^2)^{1/2}$ and note that, for $(x, t) \in \mathcal{D}(c, R)$, we have $r \in [0, \bar{R}]$ where $\bar{R} = \sqrt{1 + R^2}$. Moreover, the function φ is decreasing convex, i.e.

$$\varphi' < 0, \quad \varphi'' \geq 0 \quad \text{on } (0, \bar{R}].$$

The computation of the derivatives of χ gives

$$\frac{\partial\chi}{\partial t} = -c(1 + R^2)^{\frac{1}{2}}\varphi', \qquad D\chi = -\varphi'\,\frac{x}{(1 + |x|^2)^{\frac{1}{2}}},$$

$$D^2\chi = -\varphi'\left(\frac{I}{(1 + |x|^2)^{\frac{1}{2}}} - \frac{x \otimes x}{(1 + |x|^2)^{\frac{3}{2}}}\right) + \varphi''\,\frac{x \otimes x}{1 + |x|^2} .$$

Substituting the derivatives in $A[\cdot]$ (see Lemma 2.1), we have

$$\begin{aligned} A[\chi](x, t) = &-c(1 + R^2)^{1/2}\varphi'(r) - K_1(1 + |x|)|\varphi'(r)|\frac{|x|}{(1 + |x|^2)^{1/2}} \\ &-K_2\left(\mathrm{Tr}\left[-\varphi'(r)\left(\frac{I}{(1 + |x|^2)^{\frac{1}{2}}} - \frac{x \otimes x}{(1 + |x|^2)^{\frac{3}{2}}}\right) + \varphi''(r)\frac{x \otimes x}{1 + |x|^2}\right]^+\right)^\alpha \end{aligned} \tag{11}$$

for all $(x,t) \in \mathcal{D}(c,R)$.

We estimate (11) from below. First

$$-K_1(1+|x|)|\varphi'(r)|\frac{|x|}{(1+|x|^2)^{\frac{1}{2}}} \le 2K_1\varphi'(r)\bar{R}, \tag{12}$$

for every $(x,t) \in \mathcal{D}(c,R)$ since $|\varphi'(r)| = -\varphi'(r) \ge 0$. Noticing that the matrices

$$X = -\varphi'(r)\left(\frac{I}{(1+|x|^2)^{\frac{1}{2}}} - \frac{x\otimes x}{(1+|x|^2)^{\frac{3}{2}}}\right) \quad \text{and} \quad Y = \varphi''(r)\frac{x\otimes x}{1+|x|^2}$$

are nonnegative and, using the inequality $(a+b)^\alpha \le a^\alpha + b^\alpha$ for $a,b \ge 0$ and $0<\alpha<1$, we obtain

$$(\mathrm{Tr}[X+Y]^+)^\alpha = (\mathrm{Tr}X + \mathrm{Tr}Y)^\alpha \le (\mathrm{Tr}X)^\alpha + (\mathrm{Tr}Y)^\alpha \le N^\alpha(-\varphi'(r))^\alpha + N^\alpha(\varphi''(r))^\alpha. \tag{13}$$

From (11), (12) and (13), it follows

$$\begin{aligned} A[\chi](x,t) &\ge -\varphi'(r)(c-2K_1)\bar{R} - K_2N^\alpha\big(-\varphi'(r)\big)^\alpha - K_2N^\alpha\big(\varphi''(r)\big)^\alpha \\ &\ge\ k(c-2K_1)\bar{R}\frac{R^\beta}{r^{k+1}} - K_2'\frac{R^{\alpha\beta}}{r^{\alpha(k+1)}} - K_3'\frac{R^{\alpha\beta}}{r^{\alpha(k+2)}} \\ &\ge\ \frac{R^\beta}{r^{k+1}}\Big(\underbrace{k(c-2K_1)\bar{R} - K_2'R^{(\alpha-1)\beta}r^{(k+1)(1-\alpha)} - K_3'R^{(\alpha-1)\beta}r^{(k+1)-\alpha(k+2)}}_{S(r)}\Big) \end{aligned} \tag{14}$$

where K_2' and K_3' are positive constants which depend only on N, K_2, k and α.

We want to choose c, β and k in order that the quantity $S(r)$ in (14) is positive. We first choose k such that the quantity $S(r)$ is nonincreasing on $[0,\bar{R}]$: since the exponent $(k+1)(1-\alpha)$ of the first term in r is already positive for every choice of $k>0$, it is enough to take

$$k \ge \frac{2\alpha-1}{1-\alpha} \quad \Longrightarrow \quad (k+1)-\alpha(k+2) \ge 0\ , \tag{15}$$

in order to ensure that the exponent of the second term in r is also positive.

We are then left to choose parameters in order to have $S(\bar{R})>0$; to do so, a necessary condition is clearly

$$c > 2K_1\ . \tag{16}$$

Using the fact that $R \le \sqrt{1+R^2} = \bar{R}$, we have

$$\begin{aligned} S(\bar{R}) &= k(c-2K_1)\bar{R} - K_2'R^{(\alpha-1)\beta}\bar{R}^{(1-\alpha)(k+1)} - K_3'R^{(\alpha-1)\beta}\bar{R}^{(k+1)-\alpha(k+2)} \\ &\ge k(c-2K_1)\bar{R} - K_2'\bar{R}^{(1-\alpha)(k+1-\beta)} - K_3'\bar{R}^{(\alpha-1)\beta+(k+1)-\alpha(k+2)}. \end{aligned}$$

Now it is clear that, if we can take the exponents of $\bar{R}$ in the two last terms strictly less than 1, then we are done since for large R, the right-hand side of the inequality would be strictly positive.

This yields the following conditions

$$(1-\alpha)(k+1-\beta) < 1, \tag{17}$$

$$(\alpha-1)\beta + (k+1) - \alpha(k+2) < 1. \tag{18}$$

We recall that $0 < \alpha < 1$; on one hand, an easy computation shows that Condition (18) is automatically satisfied when (17) holds. On the other hand, Condition (17) holds if we choose β such that

$$k > \beta > k - \frac{\alpha}{1-\alpha}. \tag{19}$$

Finally, we can fix all the constants in order to fulfill (15), (16) and (19) and thus all the required properties of χ are satisfied. □

In fact, a close look at the proof of Theorem 2.1 shows that the existence of the "friendly giants" χ_R allows to get some local estimate on the difference between two solutions. It leads to the following kind of stability result which will be useful later.

Proposition 2.1 *Let $(F_\varepsilon)_{\varepsilon>0}$ a family of continuous functions satisfying assumptions **(H1)** and **(H2)** uniformly with respect to ε. Assume that there exists continuous viscosity solutions u^ε and v^ε of equation (4) with F replaced by F_ε such that $(u^\varepsilon - v^\varepsilon)(\cdot,0)$ converges locally uniformly to 0 in $I\!R^N$. Then $u^\varepsilon - v^\varepsilon$ converges locally uniformly to 0 in $I\!R^N \times [0,T]$.*

Proof of Proposition 2.1. Since the general case will follow by iterating in time, we only prove the result for $t \leq 1/2c$ recalling that c is the constant appearing in the definition of the "friendly giants" χ_R (see (6)).

We first observe that, for any $R > 0$, the function $u^\varepsilon - \sup_{B(0,R)}\{(u^\varepsilon - v^\varepsilon)(\cdot,0)\}$ is still a solution of (4) with F replaced by F_ε and that we have

$$u^\varepsilon(\cdot,0) - \sup_{B(0,R)}\{(u^\varepsilon - v^\varepsilon)(\cdot,0)\} \leq v^\varepsilon(\cdot,0) \quad \text{in } B(0,R)\ .$$

Since F_ε satisfies **(H2)** independently of ε, we get from inequality (8) that, for all $(x,t) \in \mathcal{D}(c,R)$,

$$u^\varepsilon(x,t) - v^\varepsilon(x,t) \leq \sup_{B(0,R)}\{(u^\varepsilon - v^\varepsilon)(\cdot,0)\} + \chi_R(x,t).$$

In order to prove that $u^\varepsilon - v^\varepsilon$ converges locally uniformly to 0, we take an arbitrary $\eta > 0$ and (x,t) in a compact subset K of $I\!R^N \times [0,1/2c]$. For R sufficiently large, we have $\chi_R(x,t) \leq \eta/2$ in K. Then, for ε sufficiently small, we have $\sup_{B(0,R)}\{(u^\varepsilon - v^\varepsilon)(\cdot,0)\} \leq \eta/2$. Therefore $u^\varepsilon(x,t) - v^\varepsilon(x,t) \leq \eta$ in K. And thus $\limsup_{\varepsilon\to0}\ \sup_{K}\,(u^\varepsilon - v^\varepsilon) \leq 0$.

By exchanging the roles of u^ε and v^ε, we obtain the lower estimate and the proof is complete. □

3 Existence result for equation (4)

In this section we use the comparison theorem to get the following existence result.

Theorem 3.1 *Suppose that* ***(H1)*** *and* ***(H2)*** *hold. Then, for every initial datum* $u_0 \in C(I\!R^N)$*, there exists a unique continuous viscosity solution of (4).*

Proof of Theorem 3.1. The uniqueness part is given by Theorem 2.1. For the existence, we set $\varphi_\varepsilon(r) := \min\{\max\{r, -1/\varepsilon\}, 1/\varepsilon\}$ for every $r \in I\!R$ and $\varepsilon > 0$ and consider truncations of pde (4) of the form

$$\frac{\partial u}{\partial t} + F_\varepsilon(x, t, Du, D^2 u) = 0, \tag{20}$$

where $F_\varepsilon = \varphi_\varepsilon \circ F$. Noticing that $\varphi_\varepsilon(a) - \varphi_\varepsilon(b) \leq \max\{a - b, 0\}$, we see that the F_ε satisfy assumptions **(H1)** and **(H2)**, uniformly with respect to ε.

We first get an existence result for (20) using Perron's method. Since $|F_\varepsilon(x, t, p, M)| \leq 1/\varepsilon$ for all (x, t, p, M), the function $\bar{u}_\varepsilon(x, t) := u_0(x) + t/\varepsilon$ (respectively $\underline{u}_\varepsilon(x, t) := u_0(x) - t/\varepsilon$) is a supersolution (respectively a subsolution) of (20). Theorem 2.1 provides a strong comparison result for (20) and therefore Perron's method applies readily giving the existence of a continuous solution u_ε of (20) with initial datum u_0.

The next step consists in deriving a local L^∞-bound for the family $(u_\varepsilon)_{\varepsilon>0}$ we built above. To do so, we use of the "friendly giants" introduced in Section 2. Let $R, c > 0$ and $\mathcal{D}(c, R)$ defined by formula (7). We set

$$C_R = 2 \sup_{B(0,R)} |u_0|, \quad K_R = 2 \sup_{\mathcal{D}(c,R)} |F(\cdot, \cdot, 0, 0)| \geq 2 \sup_{\mathcal{D}(c,R)} |F_\varepsilon(\cdot, \cdot, 0, 0)|,$$

and we consider the functions $(x, t) \mapsto -C_R - K_R t - \chi_R(x, t)$ and $(x, t) \mapsto C_R + K_R t + \chi_R(x, t)$ in $\mathcal{D}(c, R)$. Tedious but straightforward computations shows that these functions are respectively sub- and supersolution of (20) in $\mathcal{D}(c, R)$ and it is clear that we have $-C_R - \chi_R(x, 0) \leq u_\varepsilon(x, 0) \leq C_R + \chi_R(x, 0)$ in $B(0, R)$. Then, easy comparison arguments show that

$$-C_R - K_R t - \chi_R \leq u_\varepsilon \leq C_R + K_R t + \chi_R \quad \text{in } \mathcal{D}(c, R)\ .$$

It follows that the family $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $D(c, R/2)$ independently of ε. Iterating in time, we get the local boundedness of $(u_\varepsilon)_{\varepsilon>0}$ in $I\!R^N \times [0, +\infty)$.

Finally we apply the "half-relaxed-limits" method which consists in introducing

$$\bar{u} = \limsup_{\varepsilon \to 0^+}{}^* u_\varepsilon \quad \text{and} \quad \underline{u} = \liminf_{\varepsilon \to 0^+}{}_* u_\varepsilon$$

which are well-defined because of the local L^∞-bound on the u_ε's. Moreover, they are both discontinuous solutions of (4) with initial datum u_0 since (F_ε) converges locally uniformly to F. The strong comparison result for (4) (Theorem 2.1) shows that $\bar{u} = \underline{u} := u$ and u is the desired continuous solution of (4) we wanted to build. $\square$

4 Uniqueness for one-dimensional equations

In this section, we provide some applications of the previous result in the case of one-dimensional quasilinear parabolic equations. We address here only uniqueness questions.

We consider the equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - (f(x,t,u,u_x))_x = 0 \quad \text{in } \mathcal{D}'(I\!R \times (0,+\infty)), \\ u(\cdot,0) = u_0 \quad \text{in } I\!R, \end{cases} \tag{21}$$

where $u_0 \in C(I\!R)$, the nonlinearity $f \in C(I\!R \times [0,+\infty) \times I\!R \times I\!R)$ and $\mathcal{D}'(I\!R \times (0,+\infty))$ is the space of distributions on $I\!R \times (0,+\infty)$. For reasons which will be clear below, we consider only cases when the solution u is in $C^1(I\!R \times (0,+\infty))$.

To state our result, we use the following assumption on f

(H3) f is locally Lipschitz continuous in $I\!R \times (0,+\infty) \times I\!R \times I\!R$ and there exists constants $C > 0$ and $0 < \alpha < 1$ such that, for any $t > 0$ and $x, y, u, v, p, q \in I\!R$, we have

$$f(x,t,u,p) - f(y,t,v,q) \leq C\left[(1+|p|+|q|)|x-y| + (1+|x|+|y|)|u-v| + \left((p-q)^+\right)^\alpha\right] .$$

Note that this assumption imply **(H1)** and **(H2)** in the one-dimensional case. Equation (21) makes sense in the space of distributions since the assumed regularity for the solution ensures that both u and $f(x,t,u,u_x)$ belong to $L^1_{\text{loc}}(I\!R \times (0,+\infty))$.

Our result is the

Theorem 4.1 *Under assumptions **(H3)**, Equation (21) has at most one solution in $C^1(I\!R \times (0,+\infty)) \cap C(I\!R \times [0,+\infty))$ for each initial datum $u_0 \in C(I\!R)$.*

Proof of Theorem 4.1. We suppose, by contradiction, that we have two solutions $u, v \in C^1(I\!R \times (0,+\infty)) \cap C(I\!R \times [0,+\infty))$ of Equation (21). For every $\varepsilon > 0$, we define

$$\tilde{u}^\varepsilon(x,t) = \int_0^x u(y,t+\varepsilon)\,dy + \int_\varepsilon^{t+\varepsilon} f(0,\tau,u(0,\tau),u_x(0,\tau))\,d\tau$$

and $\tilde{v}^\varepsilon$ in the same way replacing u by v in the right-hand side of this equality.

By standard arguments in the theory of distributions, one proves easily that $\tilde{u}^\varepsilon$ and $\tilde{v}^\varepsilon$ are classical solutions of

$$\frac{\partial \omega}{\partial t} - f(x,t+\varepsilon,\omega_x,\omega_{xx}) = 0 \quad \text{in } I\!R \times (0,+\infty) . \tag{22}$$

The functions $f(\cdot,\cdot+\varepsilon,\cdot,\cdot)$ satisfy assumptions **(H1)** and **(H2)** uniformly in ε, and

$$(\tilde{u}^\varepsilon - \tilde{v}^\varepsilon)(x,0) = \int_0^x (u(y,\varepsilon) - v(y,\varepsilon))dy$$

converges locally uniformly to 0 in $\mathbb{R}$ since $u(x,0) = v(x,0) = u_0(x)$ in $\mathbb{R}$. Therefore, by applying Proposition 2.1, we deduce that $\tilde{u}^\varepsilon - \tilde{v}^\varepsilon$ converges to 0 locally uniformly in $\mathbb{R} \times [0,+\infty)$.

It follows, on one hand, that the integral

$$\int_0^t [f(0,\tau,u(0,\tau),u_x(0,\tau)) - f(0,\tau,v(0,\tau),v_x(0,\tau))]d\tau$$

is well-defined (notice that it was not the case a priori for $\int_0^t f(0,\tau,u(0,\tau),u_x(0,\tau))d\tau$ and $\int_0^t f(0,\tau,v(0,\tau),v_x(0,\tau))d\tau$) and letting ε go to 0, we obtain that, for all $(x,t) \in \mathbb{R} \times [0,+\infty)$,

$$\int_0^x (u-v)(y,t)dy + \int_0^t [f(0,\tau,u(0,\tau),u_x(0,\tau)) - f(0,\tau,v(0,\tau),v_x(0,\tau))]d\tau = 0.$$

To prove the result, we have just to differentiate this equality with respect to x. $\square$

Remark 4.1 : It would be very interesting to be able to prove the above result by assuming only the solutions to be in $W^{1,\infty}_{\mathrm{loc}}$. The difficulty to do that would be in the integration of equation (21) to get equation (22). Only few results exists in this direction and mainly for first-order equations (cf. Corrias [10]).

A key application of the above result concerns the mean curvature equation for graphs in $\mathbb{R}$.

Theorem 4.2 *The mean curvature equation for graphs (2) has a unique smooth solution* $u \in C(\mathbb{R} \times [0,+\infty)) \cap C^\infty(\mathbb{R} \times (0,+\infty))$ *for every initial datum* $u_0 \in C(\mathbb{R})$.

Proof of Theorem 4.2. The existence part of the theorem comes from the result of Ecker and Huisken [13] (see also [6]). The uniqueness part is an immediate consequence of Theorem 4.1 by taking $f(p) = \arctan(p)$. In this case (21) reads exactly (2) and, since the function arctan is a smooth nondecreasing bounded function which lies in $W^{1,\infty}(\mathbb{R})$, thus in $C^{0,\alpha}(\mathbb{R})$ for every $\alpha \in (0,1)$, **(H3)** holds and therefore Theorem 4.1 applies. $\square$

5 Application to geometrical motions in the plane

We consider, in this section, applications for the mean curvature motion of graphs in the plane $\mathbb{R}^2$. We first recall briefly some basic facts about the level-set approach in the case of the mean curvature motion.

In this framework, the graph of the initial datum $u_0 \in C(\mathbb{R})$ of (2) is represented as the hypersurface $\Gamma_0 = \{(x,y) \in \mathbb{R}^2 : u_0(x) = y\}$ in the plane. We also define $\Omega_0 =$

$\{(x,y) \in I\!R^2 : u_0(x) < y\}$ and we take any uniformly continuous function $v_0 : I\!R^2 \to I\!R$ such that

$$\Gamma_0 = \{(x,y) \in I\!R^2 : v_0(x,y) = 0\} \quad \text{and} \quad \Omega_0 = \{(x,y) \in I\!R^2 : v_0(x,y) > 0\}. \tag{23}$$

If $\mathcal{X}$ is the space of functions $w : I\!R^2 \times (0,+\infty) \to I\!R$ which are uniformly continuous in $I\!R^2 \times (0,+\infty)$ for all $T > 0$, by results of Evans and Spruck [14] and Chen, Giga and Goto [8], there exists a unique viscosity solution $v \in \mathcal{X}$ of the geometrical equation

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{1}{v_x^2 + v_y^2}(v_{xx}v_y^2 - 2v_{xy}v_xv_y + v_{yy}v_x^2) = 0 & \text{in } I\!R^2 \times (0,+\infty), \\ v(\cdot,\cdot,0) = v_0 \quad \text{in } I\!R^2. \end{cases}$$

Moreover, if we define, for every $t \geq 0$,

$$\Gamma_t = \{(x,y) \in I\!R^2 : v(x,y,t) = 0\} \quad \text{and} \quad \Omega_t = \{(x,y) \in I\!R^2 : v(x,y,t) > 0\}, \tag{24}$$

then the sets $(\Gamma_t)_{t\geq 0}$ and $(\Omega_t)_{t\geq 0}$ depend only on Γ_0 and Ω_0 but not on the choice of their representation through v_0.

The family $(\Gamma_t)_{t\geq 0}$ is called *the generalized evolution by mean curvature of the graph* Γ_0. A natural issue is the connection between this generalized evolution and the classical motion by mean curvature. We recall that in general Γ_t is just defined as the 0-level set of a continuous function and therefore it may be nonsmooth and even fatten.

In our context, we have

Theorem 5.1 *If $u_0 \in C(I\!R)$, then, for every $t \geq 0$, the set Γ_t is a entire smooth graph, namely*

$$\Gamma_t = \{(x,y) \in I\!R^2 : y = u(x,t)\},$$

where u is the unique smooth solution of (2) with initial datum u_0. Moreover, the evolution of Γ_t agrees with the classical motion by mean curvature in the sense of differential geometry.

Proof of Theorem 5.1. From [6], we know that, if we start with an hypersurface Γ_0 which is an entire continuous graph in $I\!R \times I\!R$, then, for every $t \geq 0$,

$$\Gamma_t = \{(x,y) \in I\!R^2 : u^-(x,t) \leq y \leq u^+(x,t)\},$$

where u^- and u^+ are respectively the minimal and the maximal (possibly discontinuous) viscosity solution of (2). In the special case of the mean curvature equation, we proved that the boundary of the front Γ_t is smooth. It follows that u^- and u^+ are smooth. Uniqueness for (2) in the class of smooth functions (see Theorem 4.2) implies that $u^- = u^+ = u$ where u is the unique solution to (2). Finally, $\Gamma_t = \text{Graph}(u(\cdot,t))$ is a smooth submanifold of $I\!R^2$ (in particular, Γ_t never fattens). In this case, the generalized evolution coincides with the classical evolution by mean curvature (see Evans and Spruck [14] and [16] for the agreement with an alternative generalized motion). □

Using the previous geometrical approach, we can state a refinement of Theorem 4.2, namely a comparison result which holds for any viscosity sub- and supersolutions of (2).

Corollary 5.1 *If u_1 (resp. u_2) is an upper-semicontinuous viscosity subsolution (respectively lower-semicontinuous supersolution) of (2) and if $u_1(x,0) \leq u_0(x) \leq u_2(x,0)$ in $I\!R$, then $u_1 \leq u_2$ in $I\!R \times [0,+\infty)$.*

Proof of Corollary 5.1. This result is an immediate consequence of a more general result in the case of the mean curvature equation: uniqueness for smooth solutions implies comparison in the class of discontinuous viscosity solutions (see [6] for a proof). To be self-contained, we provide a short proof which emphasizes the main ideas of the proof.

By Theorem 6.1 in [6], since u_1 and u_2 are respectively an usc viscosity subsolution and a lsc supersolution of (2), we have

$$\mathrm{Graph}(u_1(\cdot,t)) \subset \{(x,y) \in I\!R^{N+1} : v(x,y,t) \leq 0\}\ ,$$

and

$$\mathrm{Graph}(u_2(\cdot,t)) \subset \{(x,y) \in I\!R^{N+1} : v(x,y,t) \geq 0\}\ .$$

Essentially, this comes from the preservation of inclusion for sets moving by their mean curvature in the level set approach (see Evans and Spruck [14]).

We recall that v is an increasing function of y but the above inclusions do not give any information about the relative position of $\mathrm{Graph}(u_1(\cdot,t))$ and $\mathrm{Graph}(u_2(\cdot,t))$ when the fronts $\Gamma_t(u_0)$ develop interior. But, thanks to Theorem 5.1, we know that $\Gamma_t(u_0)$ is exactly the graph of $u(\cdot,t)$, where u is the unique smooth solution of (2) with initial datum u_0. The front does not fatten and is the boundary of $\Omega_t(u_0)$. It follows that

$$u_1(\cdot,t) \leq u(\cdot,t) \leq u_2(\cdot,t) \quad \text{in } I\!R^N\ ,$$

which ends the proof. □

We conclude this section with an extension of the previous results to some more general equations associated to more general geometric motions. We consider

$$\begin{cases} \dfrac{\partial u}{\partial t} - \left(f(u_x)\right)_x = 0 & \text{in } I\!R \times (0,+\infty), \\ u(\cdot,0) = u_0 & \text{in } I\!R, \end{cases} \tag{25}$$

where $f \in C^1(I\!R)$.

Theorem 5.2 *Let $u_0 \in W^{1,\infty}_{\mathrm{loc}}(I\!R)$. Suppose that*

$$f \in C^1(I\!R) \quad \text{and} \quad 0 < f'(p) \leq \frac{C}{1+p^2} \quad \text{for every } p \in I\!R. \tag{26}$$

Then (25) admits a unique solution $u \in C^2(I\!R \times (0,+\infty)) \cap C(I\!R \times [0,+\infty))$. The generalized evolution of $\Gamma_0 = \mathrm{Graph}(u_0)$ is $\Gamma_t = \mathrm{Graph}(u(\cdot,t))$ for $t \geq 0$. It evolves with normal velocity equal to

$$\mathcal{V}((x,y),t) = f'(-\mathrm{cotan}(\theta)) \frac{\kappa}{\sin^2(\theta)}, \tag{27}$$

where θ and κ denote respectively the angle between the y-axis and the normal outward vector and the curvature to Γ_t at the point (x,y).

Proof of Theorem 5.2. We consider as above an uniformly continuous function $v_0 : I\!R^2 \to I\!R$ such that $\{v_0 = 0\} = \Gamma_0$ and $\{v_0 > 0\} = \Omega_0 := \{y > u_0(x)\}$. Following [6, Section 4], if (26) holds, then

$$\begin{cases} \dfrac{\partial v}{\partial t} - f'\left(-\dfrac{v_x}{v_y}\right)(v_{xx} - 2v_{xy}\left(\dfrac{v_x}{v_y}\right) + v_{yy}\left(\dfrac{v_x}{v_y}\right)^2) = 0 & \text{in } I\!R^2 \times (0,+\infty), \\ v(\cdot,\cdot,0) = v_0 & \text{in } I\!R^2 \end{cases} \tag{28}$$

admits a unique solution $v \in \mathcal{X}$. The level set approach applies for the generalized evolution $(\Gamma_t)_{t\geq 0}$ of Γ_0 and evolves formally with normal velocity given by (27).

On a other hand, using Chou and Kwong [9], we learn from [6] that, again because of (26), the boundary of Γ_t is made of the graphs of two smooth solutions of (25), namely u^+ and u^-. Noticing that (26) implies **(H3)**, we obtain that (25) has a unique smooth solution; thus $u^+ = u^- := u$ and $\Gamma_t = \mathrm{Graph}(u(\cdot,t))$. Finally, note that, since Γ_t is smooth, (27) holds in a classical sense. □

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