Master 2 Mathématiques et Applications–PUF Ho Chi Minh Ville–2009/10
"Viscosity solutions, HJ Equations and Control"–O.Ley (INSA de Rennes)

Exam – February 2010 – 3h

– *Written-by-hands documents are allowed.*
– *Printed documents, computers, cellular phones are forbiden.*
– *The text is composed of 4 pages.*
– *Do not worry about the length of the text. It is not necessary to answer all questions to have the maximum mark.*
– *Answer seriously, rigorously and clearly the questions you choose to work.*
– *You may use* without proof *the results which were proven in the lecture.*
– *Exercises in Part I are independent. Part II, III and IV can be treated independently in any order.*
– *For the correction see my webpage:* `http://www.lmpt.univ-tours.fr/~ley` *(teaching)*

Notations: In $\mathbb{R}^N$, we consider the classical Euclidean inner product

$$\langle x, y\rangle = \sum_{i=1}^{N} x_i y_i \quad \text{for all } x = (x_1, x_2, \ldots, x_N), y = (y_1, y_2, \ldots, y_N) \in \mathbb{R}^N.$$

The Euclidean norm is written $|\cdot|$ or $||\cdot||$:

$$|x| = ||x|| = \langle x, x\rangle^{1/2} = \sqrt{\sum_{i=1}^{N} x_i^2}.$$

I. Preliminaries exercises

I.1. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a *bounded* continuous function. For any $\alpha > 0$, we define

$$M_\alpha = \sup_{x\in\mathbb{R}^N} \{f(x) - \alpha||x||\} \quad \text{and} \quad M = \sup_{x\in\mathbb{R}^N} \{f(x)\}.$$

Prove that $\lim\limits_{\alpha\to 0} M_\alpha = M$.

I.2. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous function. Suppose that f is *not* differentiable at $x_0 \in \mathbb{R}^N$. Prove that either $D^+f(x_0) = \emptyset$ or $D^-f(x_0) = \emptyset$.

I.3. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a K-Lipschitz continuous function ($K > 0$), that is,

$$|f(x) - f(y)| \le K||x - y|| \quad \text{for all } x, y \in \mathbb{R}^N.$$

Let $\bar{x} \in \mathbb{R}^N$. Prove that: if $\bar{p} \in D^+f(\bar{x})$, then $||\bar{p}|| \le K$.

I.4. Let $\psi : \mathbb{R}^N \times \mathbb{R}^P \to \mathbb{R}$ be a C^1 function. Let C be a compact subset of $\mathbb{R}^P$. We define

$$\varphi(x) = \inf_{y \in C} \psi(x, y).$$

Let $x_0 \in \mathbb{R}^N$ $y_0 \in C$ be such that $\varphi(x_0) = \psi(x_0, y_0)$. Prove that

$$D_x\psi(x_0, y_0) \in D^+\varphi(x_0)$$

($D_x\psi$ denotes the gradient of ψ with respect to the x variable).

II. The distance function is a viscosity solution to the Eikonal equation

Let $\Omega \subset \mathbb{R}^N$ be an *open bounded* subset. We consider the Eikonal equation

$$\begin{cases} ||Du|| = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The subset $\partial\Omega = \overline{\Omega} \setminus \mathrm{int}(\Omega)$ is the boundary of Ω, $\overline{\Omega}$ is its closure and $\mathrm{int}(\Omega)$ is its interior. Notice that $\partial\Omega$ and $\overline{\Omega}$ are compact subset of $\mathbb{R}^N$.

We introduce the distance function to $\partial\Omega$:

$$d_{\partial\Omega}(x) = \inf_{y \in \partial\Omega} ||y - x||.$$

II.1. Show that $d_{\partial\Omega}$ is 1-Lipschitz continuous in $\mathbb{R}^N$ and that, $d_{\partial\Omega}(x) = 0$ if and only if $x \in \partial\Omega$.

II.2. Let $x \in \Omega$ and $p \in D^+d_{\partial\Omega}(x)$. Prove that $||p|| \leq 1$ and conclude that $d_{\partial\Omega}$ is a viscosity subsolution of (1) in Ω.

II.3. Let $x_0 \in \Omega$. We distinguish two cases.
II.3.1. Case 1: there exists a *unique* $y_0 \in \partial\Omega$ such that $d_{\partial\Omega}(x_0) = ||x_0 - y_0||$. Prove that $d_{\partial\Omega}$ is then differentiable at x_0 and that $||Dd_{\partial\Omega}(x_0)|| = 1$.
[You may use, without proof, some results of an exercise solved during the lecture.]
II.3.2. Case 2: assume that y_0 in Case 1 is not unique. Prove that $(x_0 - y_0)/||x_0 - y_0|| \in D^+d_{\partial\Omega}(x_0)$. Conclude that, either $d_{\partial\Omega}(x_0)$ is differentiable at x_0 with $||Dd_{\partial\Omega}(x_0)|| = 1$, or $d_{\partial\Omega}$ is not differentiable at x_0 and $D^-d_{\partial\Omega}(x_0) = \emptyset$. Conclude that $d_{\partial\Omega}$ is a supersolution of (1).
[You may use I.4 and I.2.]
II.3.3. Draw a quick picture which illustrates Case 1 and Case 2 in II.3.1 and II.3.2.

II.4. Conclude that $d_{\partial\Omega}$ is a viscosity solution of (1).

III. The distance function is the unique viscosity solution of (1).

We use the notations and the definitions of Section II.

III.1. Write the Hamiltonian $H(x, r, p)$ associated with (1). Does this Hamiltonian satisfy (H1)? (H2)? ((H1) and (H2) are the assumptions introduced in the Section "Uniqueness" in the lecture).

It follows that we cannot apply directly the uniqueness theorem (stationary case) proven in the lecture.

III.2. We make the change of function $v = \Phi(u) = -\exp(-u)$ in (1). Prove that v satisfies a new Hamilton-Jacobi equation

$$\begin{cases} F(v, Dv) = 0 & \text{in } \Omega, \\ v = c & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Give $F(r, p)$ and the constant c.
[You can use the result of the lecture about the change of functions in Hamilton-Jacobi equations without proof. But you have to justify that Φ is an admissible change of function.]

III.3. Prove that the new Hamiltonian F satisfies (H1) and (H2) and conclude that (2) (and therefore (1)) has a unique viscosity solution.

IV. Optimal control

You can apply, without proof, all the results about optimal control which were proven in the lecture.

We consider the controlled ordinary differential equation

$$\begin{cases} \dot{X}_x(s) = \alpha(s) & \text{for } 0 < s \le T, \\ X_x(0) = x \in \mathbb{R}, \end{cases} \tag{3}$$

where the control $\alpha \in L^\infty([0, T])$ is such that

$$\alpha(s) \in [0, 1] \quad \text{for almost all } s \in [0, T].$$

We consider the control problem of minimizing, among all admissible control, the cost

$$J(x, t, \alpha(\cdot)) = \int_0^t ds + u_0(X_x(t)),$$

where $x \in \mathbb{R}$, $t \in [0, T]$ and u_0 is a given real-valued bounded Lipschitz continuous function. We introduce as usual the value function

$$V(x, t) = \inf_{\alpha \in L^\infty([0,T]), |\alpha| \le 1 \text{ a.e.}} \{J(x, t, \alpha(\cdot))\}$$

IV.1. Write the general form of the solution of (3). What is the trajectory $X_x(t)$ when $T = 1$, $\alpha(s) = 1$ for $0 \leq s \leq 1/2$ and $\alpha(s) = -s$ for $1/2 < s \leq 1$?

IV.2. Find a Hamilton-Jacobi-Bellman equation such that the value function V is the unique viscosity solution.

IV.3. Check (formally) that $W(x,t) = V(x,t) - t$ is the unique viscosity solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} + \left|\dfrac{\partial u}{\partial x}\right| = 0 & \text{in } \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}. \end{cases}$$

Deduce a formula for $V(x,t)$.

IV.4. Let $u_0(x) = 1$. What is $V(x,t)$ in this case?

IV.5. Let $u_0(x) = \frac{1}{1+x^2}$. What is $V(x,t)$ in this case?

IV.6. Let $u_0(x) = \sin(x)$. Prove that $V(x,t) = -1 + t$ if $t \geq \pi$.

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