Geometric flows and Bernoulli problem

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Statement of the problem and motivation

The aim of the work [CL] is to study nonlocal geometric flows $(\Omega(t))_{t\geq 0}$. For the moment, suppose that all sets considered are smooth enough to give sense to our calculations. For all $t \geq 0$, $\Omega(t)$ is a bounded subset of $I\!R^N$ whose boundary $\partial\Omega(t)$ evolves with a normal velocity of the type

$$\mathcal{V}_{t,x} = F(\nu_x, H_x) + \lambda h(x, \Omega(t)) \quad \text{for every } x \in \partial\Omega(t), \tag{1}$$

where $\lambda \geq 0$, ν_x is the outward unit normal to $\Omega(t)$ at x, H_x is the curvature matrix of $\partial\Omega(t)$ at x (nonpositive for convex sets), F is continuous and elliptic, i.e. nondecreasing with respect to the curvature matrix. The nonlocal term h is of Hele-Shaw type:

$$h(x, \Omega(t)) = |\nabla u(x)|^\beta \quad \text{with } \beta = 1 \text{ or } 2, \tag{2}$$

where u is the solution of an auxiliary partial differential equation

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega(t)\backslash S, \\ u = 1 & \text{on} \quad \partial S, \\ u = 0 & \text{on} \quad \partial\Omega(t). \end{cases} \tag{3}$$

The set S is a fixed source with C^2 boundary and we always assume $S \subset\subset \Omega(t)$.

In our talk, for simplicity, we will focus on two model cases:

$$\mathcal{V}_{t,x} = -1 + \lambda|\nabla u(x)|^2 \tag{4}$$

and

$$\mathcal{V}_{t,x} = H_x + \lambda|\nabla u(x)|^2. \tag{5}$$

The motivation to study such problems comes from the numerical work of Allaire, Jouve and Toader [AJT] in shape optimization. They use formally a gradient method for the minimization of an objective function $J(\Omega)$ where Ω is a subset of $\mathbb{R}^N$.

Let us describe briefly their approach in the case related to the above velocity (4). Consider the problem of minimizing the capacity of a set under volume constraint:

$$\min_{S\subset\subset\Omega\subset\subset\mathbb{R}^N}\{\mathrm{cap}(\Omega) \text{ with } \mathrm{vol}(\Omega)=\text{constant}\}, \tag{6}$$

where

$$\mathrm{cap}(\Omega)=\int_{\Omega\setminus S}|\nabla u(x)|^2dx, \qquad \mathrm{vol}(\Omega)=\int_{\Omega\setminus S}dx$$

and u is the solution of (3). For any local diffeomorphism θ which maps Ω to $\theta(\Omega)$, we can compute the shape derivatives with respect to θ of the two previous quantities. By Hadamard formulas, we get

$$\mathrm{cap}'(\Omega)(\theta)=-\int_{\partial\Omega}|\nabla u(x)|^2\langle\theta(x),\nu_x\rangle d\sigma \quad\text{and}\quad \mathrm{vol}'(\Omega)(\theta)=\int_{\partial\Omega}\langle\theta(x),\nu_x\rangle d\sigma,$$

where $\langle\cdot,\cdot\rangle$ is the usual euclidean inner product and $d\sigma$ is the induced measure on $\partial\Omega$. Writing the necessary condition of optimality, there exists a Lagrange multiplier $\Lambda\geq 0$ such that

$$\mathrm{cap}'(\Omega)(\theta)+\Lambda\mathrm{vol}'(\Omega)(\theta)=0.$$

If we set

$$J_\lambda(\Omega)=\mathrm{vol}(\Omega)+\lambda\mathrm{cap}(\Omega),$$

and choose $\theta=-1+\lambda|\nabla u(x)|^2$ as in (4), then, at least formally, we get

$$J'_\lambda(\Omega)(\theta)=-\int_{\partial\Omega}(-1+\lambda|\nabla u(x)|^2)^2d\sigma\leq 0.$$

Therefore $\theta=-1+\lambda|\nabla u(x)|^2$ appears as a descent direction for the optimization problem (6). The method used in [AJT] to solve (6) is now clear: they fix an initial set Ω_0, consider the evolution $(\Omega_t)_{t\geq 0}$ with normal velocity (4) and compute the limit of Ω_t as $t\to+\infty$ which is the candidate minimizer to (6).

We end with two important remarks. At first, problem (6) is equivalent to the well-known Bernoulli exterior free boundary problem (see [FR] for a survey):

$$\text{Find a set } S\subset\subset\Omega\subset\subset\mathbb{R}^N \text{such that } |\nabla u(x)|=\frac{1}{\sqrt{\lambda}} \text{ for all } x\in\partial\Omega. \tag{7}$$

Secondly, (5) appears when considering the previous problem with perimeter constraint instead of volume constraint:

$$\min_{S \subset\subset \Omega \subset\subset \boldsymbol{R}^N} \{\operatorname{cap}(\Omega) \text{ with } \operatorname{per}(\Omega) = \text{constant}\}, \tag{8}$$

where

$$\operatorname{per}(\Omega) = \int_{\partial\Omega} d\sigma.$$

In this case, for any local diffeomorphism θ,

$$\operatorname{per}'(\Omega)(\theta) = -\int_{\partial\Omega} H_x \langle \theta(x), \nu_x \rangle d\sigma$$

and $\theta = H_x + \lambda |\nabla u(x)|^2$ as in (5) looks as a descent direction for

$$J_\lambda(\Omega) = \operatorname{per}(\Omega) + \lambda \operatorname{cap}(\Omega).$$

Definition of solutions

As we said at the beginning, this approach relies on the assumption that all sets are smooth enough to give sense of our computations. But in reality, even for nicer velocities (as mean curvature for instance), the evolutions face a lack of regularity and singularities occur in finite time.

We intend to make the approach of [AJT] as rigorous as possible by defining generalized solutions for (4) and (5) widely inspired from the theory of viscosity solutions. Before describing our method and stating our results, let us recall some previous works on evolutions with prescribed normal velocity close to ours.

Following the numerical work of Osher and Sethian, a breakthrough was made by the articles of Chen, Giga and Goto [CGG] and Evans and Spruck [ES] in the case of local evolutions. They described the evolution as the level set of the solution of an auxiliary pde, the level set equation. This equation is solved in the sense of viscosity solutions (see [CIL]). This powerful method leads to plenty of results and was developed, in addition to the quoted mathematicians, by Barles, Ishii, Ohnuma, Sato, Soner, Souganidis and many others. We refer to Giga [G] for an overview.

When dealing with nonlocal velocities, it is not easy to write and study the level set equation. Some results in this direction were obtained recently by Kim, Slepcev and Da Lio. Our method does not use the level set approach.

Instead, we use generalized solutions which are kind of "geometric viscosity solutions" and were introduced by Cardaliaguet [C1], [C2]. Next, in [CR], these solutions were used to solve Hele-Shaw problem. The main novelty in our work is that we can deal with nonlocal Hele-Shaw terms like (2) and mean curvature as in (5) at the same time.

Before giving the definition of generalized solutions to (1), we need to introduce some notations.

Our evolution $(\Omega_t)_{t\geq 0}$ will be described by a *tube* $\mathcal{K}$ which is a subset of $\mathbb{R}^+ \times \mathbb{R}^N$ such that $\overline{\mathcal{K}} \cap ([0,T]\times \mathbb{R}^N)$ is a compact subset of $\mathbb{R}^{N+1}$ for any $T \geq 0$. We recover the desired evolution at time t by setting $\Omega(t) := \mathcal{K}(t)$. We denote by $\widehat{\mathcal{K}} = \overline{\mathbb{R}^N - \mathcal{K}}$ the *exterior of the tube.*

If $\mathcal{K}$ is a C^1 tube (i.e. a tube whose boundary has at least C^1 regularity) then, in a natural way, at any point $(t,x)\in\partial\mathcal{K}$, the *normal velocity* $\mathcal{V}^{\mathcal{K}}_{t,x}$ to $\mathcal{K}(t)$ at x is defined by

$$\mathcal{V}^{\mathcal{K}}_{(t,x)} = -\frac{\nu_t}{|\nu_x|}. \tag{9}$$

A *regular tube* $\mathcal{K}$ is a tube with a non empty interior whose boundary has at least C^1 regularity, such that at any point $(t,x)\in\partial\mathcal{K}$, the normal velocity is finite:

$$\mathcal{V}^{\mathcal{K}}_{(t,x)} < \infty \quad \Longleftrightarrow \quad \nu_x \neq 0.$$

The above regularity assumption is generalized to nonsmooth tube as follows: we say that a tube $\mathcal{K}$ is *left lower semi-continuous* if

$$\forall t>0,\ \forall x\in\mathcal{K}(t),\ \text{if } t_n\to t^-,\ \ \exists x_n\in\mathcal{K}(t_n) \text{ such that } x_n\to x\ .$$

A C^1 regular tube $\mathcal{K}_r$ is *externally tangent* to a tube $\mathcal{K}$ at $(t,x)\in\mathcal{K}$ if

$$\mathcal{K}\subset\mathcal{K}_r \text{ and } (t,x)\in\partial\mathcal{K}_r\ .$$

It is *internally tangent* to $\mathcal{K}$ at $(t,x)\in\widehat{\mathcal{K}}$ if

$$\mathcal{K}_r\subset\mathcal{K} \text{ and } (t,x)\in\partial\mathcal{K}_r\ .$$

The reason to introduce externally and internally tangent tubes is clear when making the analogy with viscosity solutions: such tubes will play the role of test-functions. With this aim, it remains to decide what regularity one has to assume for test-tubes.

Looking at (1), we see that the local term (which depends only on the curvature of $\Omega(t)$ at x) has a sense as soon as $\partial\Omega$ is C^2 in a neighborhood of (t,x). On the other hand, from classical pde theory, we know that we has to assume that $\partial\Omega(t)$ is $C^{1,1}$ to solve (3) and compute the nonlocal part in (1). Therefore, we will say that $K^s_{t,x}$ is a *smooth test-tube* at (t,x) if $(t,x) \in \partial K^s_{t,x}$ and $K^s_{t,x}$ is a $C^{1,1}$ regular tube with a C^2 boundary in a neighborhood of (t,x).

We are now ready to give the definition of generalized solutions:

Definition *(Generalized solutions) Let $\mathcal{K}$ be a tube and $S \subset\subset K_0 \subset\subset I\!\!R^N$ be an initial set.*

1. *$\mathcal{K}$ is* a viscosity subsolution *to (1) if $\mathcal{K}$ is left lower semi-continuous, $S \subset\subset \mathcal{K}(t)$ for any t, and if, for any smooth test-tube $K^s_{t,x}$ externally tangent to $\mathcal{K}$ at (t,x) with $t>0$, we have*

$$\mathcal{V}^{\mathcal{K}^s_{t,x}}_{(t,x)} \leq F(\nu_x, H^{\mathcal{K}^s_{t,x}}_x) + \lambda h(x, \mathcal{K}^s_{t,x}(t)),$$

   where ν_x is the spatial component of the outward unit normal and $H^{\mathcal{K}^s_{t,x}}_x$ is the curvature matrix to $\mathcal{K}^s_{t,x}(t)$ at (t,x).

   We say that $\mathcal{K}$ is a subsolution to (1) with initial position K_0 if $\mathcal{K}$ is a subsolution and if $\overline{\mathcal{K}}(0) \subset \overline{K_0}$.

2. *$\mathcal{K}$ is a* viscosity supersolution *to (1) if $\widehat{\mathcal{K}}$ is left lower semi-continuous, $S \subset\subset \mathcal{K}(t)$ for any t, and if, for any smooth test-tube $K^s_{t,x}$ internally tangent to $\mathcal{K}$ at (t,x) with $t>0$, we have*

$$\mathcal{V}^{\mathcal{K}^s_{t,x}}_{(t,x)} \geq F(\nu_x, H^{\mathcal{K}^s_{t,x}}_x) + \lambda h(x, \mathcal{K}^s_{t,x}(t)).$$

   We say that $\mathcal{K}$ is a supersolution to (1) with initial position K_0 if $\mathcal{K}$ is a supersolution and if $\widehat{\mathcal{K}}(0) \subset \overline{I\!\!R^N \backslash K_0}$.

3. *Finally, we say that a tube $\mathcal{K}$ is a* viscosity solution *to (1) (with initial position K_0) if $\mathcal{K}$ is a sub- and a supersolution.*

Statement of the results

Our main result is the following preservation of inclusion:

Theorem *(Inclusion principle) Let* $T > 0$ *and* $0 < \lambda_1 < \lambda_2$ *be fixed. Suppose* $\mathcal{K}_1$ *(respectively* $\mathcal{K}_2$*) is a subsolution (respectively a supersolution) to (4) or (5) with* $\lambda = \lambda_1$ *(respectively with* $\lambda = \lambda_2$*) on the time interval* $[0, T)$*. If*

$$\overline{\mathcal{K}_1}(0) \cap \widehat{\mathcal{K}_2}(0) = \emptyset \ ,$$

then

$$\forall t \in [0, T), \qquad \overline{\mathcal{K}_1}(t) \cap \widehat{\mathcal{K}_2}(t) = \emptyset \ .$$

A sketch of the proof will be given in the talk. This result corresponds to a comparison result for viscosity solutions. It implies existence and uniqueness of solutions.

Theorem *(Existence) Let* $S \subset\subset K_0 \subset\subset I\!\!R^N$*. There exists at least one solution to (4) (or (5)) with initial position* K_0*. More precisely, there exists a largest solution denoted by* $S(K_0)$ *which contains all the subsolution* $\mathcal{K}$ *such that* $\overline{\mathcal{K}} \subset \overline{K_0}$ *and there exists a smallest solution denoted by* $s(K_0)$ *which is contained in all the supersolution* $\mathcal{K}$ *such that* $\overline{\mathcal{K}} \supset \overline{K_0}$*.*

We continue by giving a first result of uniqueness:

Theorem *(Generic uniqueness) Let* $(K_0^\lambda)_{\lambda \in (0,+\infty)}$ *be a family of initial positions such that, if* $\lambda' < \lambda$*, then* $K_0^{\lambda'} \subset K_0^\lambda$ *and* $\partial K_0^{\lambda'} \cap \partial K_0^\lambda = \emptyset$*. Let* $s(K_0^\lambda)$ *(respectively* $S(K_0^\lambda)$*) be the smallest (respectively biggest) solution for (4) (or (5)) with* λ *and initial position* K_0^λ*. We have uniqueness in the following sense: there exists a countable subset* I *of* $(0, +\infty)$ *such that*

$$\overline{s(K_0^\lambda)} = S(K_0^\lambda) \quad \text{for all } \lambda \in (0, +\infty) \backslash I.$$

Now, we turn to the asymptotic behaviour of $\mathcal{K}(t)$ as $t \to +\infty$ as announced. From now on, we consider the evolution problem with velocity given by (4). As we said above, this problem is related to the Bernoulli exterior free boundary problem (7). We start to give a definition of generalized solutions to (7) (or equivalently (6)).

Definition *(Generalized solutions for the Bernoulli problem) A set $\Omega \subset I\!\!R^N$ is a solution to (7) is the constant tube $\mathcal{K} = [0, +\infty) \times \Omega$ is a solution to (4).*

There are different notions of weak solutions for (7). The one we give here is the most suitable for our purpose.

Theorem *(Existence and Uniqueness for the Bernoulli problem) Suppose that the source S is strictly starshaped. Then for any $\lambda > 0$ there exists a unique solution Ω_λ to (7).*

This result was first proved by Tepper [T]. We conclude with

Theorem *(Asymptotic behaviour) Let $\lambda > 0$ and suppose that the source S is strictly starshaped and consider $S \subset\subset K_0 \subset\subset I\!\!R^N$. Then every solution $\mathcal{K}$ to (4) with initial position K_0 converges (for the Hausdorff metric) to the unique solution of (7) as $t \to +\infty$.*

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